Chapter 3 Basic Dynamics

3.1 Introduction

The atmosphere is a shallow envelope of compressible gas surrounding an approximately spherical, rotating planet. The equations of motion in a rotating frame for such a gas are well known, but in their most general form they are far more complicated than necessary or desirable for application to the large- and medium-scale meteorological phenomena considered in this book. Scale analysis, involving an investigation of the relative orders of magnitude of the various terms in the relevant equations, shows that several simplifications to the equations can be made. In particular, the vertical momentum equation can be replaced by hydrostatic balance, the Coriolis force associated with the horizontal component of the earth's rotation vector can be neglected, and the distance r from any point in the atmosphere to the center of the earth can be replaced by a mean radius a. The resulting approximate set of equations is called the *primitive equations.*

3.1.1 The Primitive Equations in Log-Pressure Coordinates on the Sphere

Although the geometric height $z^* \equiv r - a$ is the most obvious choice of vertical coordinate, the primitive equations take slightly simpler forms when other vertical coordinates are used, and throughout most of this book we shall use the "log-pressure" coordinate

$$z \equiv -H \ln(p/p_s) \tag{3.1.1}$$

introduced in Eq. (1.1.8). (The inverse relation

$$p = p_s e^{-z/H}, \tag{3.1.2}$$

should be noted.) Using z as the vertical coordinate and spherical coordinates in the horizontal, the primitive equations take the following form:

$$\frac{Du}{Dt} - \left(f + \frac{u \tan \phi}{a}\right) v + \frac{\Phi_\lambda}{a \cos \phi} = X, \tag{3.1.3a}$$

$$\frac{Dv}{Dt} + \left(f + \frac{u \tan \phi}{a}\right) u + \frac{\Phi_\phi}{a} = Y, \tag{3.1.3b}$$

$$\Phi_z = H^{-1} R\theta\, e^{-\kappa z/H}, \tag{3.1.3c}$$

$$\frac{[u_\lambda + (v \cos \phi)_\phi]}{a \cos \phi} + \frac{(\rho_0 w)_z}{\rho_0} = 0, \tag{3.1.3d}$$

$$\frac{D\theta}{Dt} = Q, \tag{3.1.3e}$$

(e.g., Holton, 1975). These express, respectively, momentum balance in the zonal and meridional directions, hydrostatic balance in the vertical, continuity of mass, and the thermodynamic relation between diabatic heating and the material rate of change of potential temperature.

In Eq. (3.1.3) Φ denotes the geopotential [see Eq. (1.1.3)], θ the potential temperature [see Eq. (1.1.9)], and the following new notation has been used:

Horizontal coordinates: (λ, ϕ) = (longitude, latitude).
"Velocity" components:

$$(u, v, w) \equiv \left[(a \cos \phi)\frac{D\lambda}{Dt}, a\frac{D\phi}{Dt}, \frac{Dz}{Dt}\right],$$

where D/Dt is the material derivative, or time rate of change following the fluid motion, whose expression in the present coordinates is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a \cos \phi}\frac{\partial}{\partial \lambda} + \frac{v}{a}\frac{\partial}{\partial \phi} + w\frac{\partial}{\partial z}.$$

Note that w is not in general equal to the geometric vertical velocity Dz^*/Dt; however, the difference is generally insignificant except near the ground.

Coriolis parameter (the vertical component of the earth's rotation vector): $f \equiv 2\Omega \sin \phi$, where $\Omega = 2\pi$ (sidereal day)$^{-1}$ = $7.292 \times 10^{-5}\ \mathrm{s}^{-1}$ is the earth's rotation rate.

Unspecified horizontal components of friction, or other nonconservative mechanical forcing: (X, Y).

Diabatic heating term: $Q \equiv (J/c_p)e^{\kappa z/H}$, where J is the diabatic heating rate per unit mass, which in the middle atmosphere equals the net *radiative* heating rate per unit mass, $-\rho_0^{-1}\,\partial F_n/\partial z$ (see Section 2.5), plus a small thermal conduction term; note that J/c_p is often expressed in units of Kelvins per day.

Basic density: $\rho_0(z) \equiv \rho_s e^{-z/H}$ where $\rho_s \equiv p_s/RT_s$. Thus $\rho_0 = p/RT_s$ by Eq. (3.1.2); some authors use p instead of ρ_0 in Eq. (3.1.3d).

Some partial derivatives with respect to λ, ϕ, and z are denoted by suffixes.

The primitive equations are frequently written using the temperature T instead of potential temperature θ, in which case Eqs. (3.1.3c, e) are replaced by

$$\Phi_z = H^{-1}RT \tag{3.1.3c$'$}$$

and

$$\frac{DT}{Dt} + \frac{\kappa T w}{H} = \frac{J}{c_p}, \tag{3.1.3e$'$}$$

respectively.

A further quantity of considerable dynamical importance is Ertel's potential vorticity P (Rossby (1940), Ertel (1942)), defined in general as $\rho^{-1}\boldsymbol{\omega}_a \cdot \nabla\theta$ where $\boldsymbol{\omega}_a$ is the absolute vorticity. Under the approximations that lead to the primitive equations, it is given by

$$\rho_0 P \equiv \theta_z\left[f - \frac{(u\cos\phi)_\phi}{a\cos\phi} + \frac{v_\lambda}{a\cos\phi}\right] - \frac{\theta_\lambda v_z}{a\cos\phi} + \frac{\theta_\phi u_z}{a} \tag{3.1.4}$$

in log-pressure coordinates. A rather lengthy calculation, starting from Eq. (3.1.3), shows that

$$\frac{DP}{Dt} = (\rho_0 a\cos\phi)^{-1}\left[-\frac{\partial(X\cos\phi,\theta)}{\partial(\phi,z)} + \frac{\partial(Y,\theta)}{\partial(\lambda,z)} - \frac{\partial(Q,v)}{\partial(\lambda,z)} + \frac{\partial(Q,m)}{\partial(\phi,z)}\right], \tag{3.1.5}$$

where

$$\frac{\partial(A,B)}{\partial(x,y)} \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$$

and

$$m \equiv a\Omega\cos^2\phi + u\cos\phi$$

is a^{-1} times the absolute zonal angular momentum per unit mass. Note that if the mechanical forcing (X, Y) and the diabatic heating Q both vanish, the right-hand side of Eq. (3.1.5) vanishes, and P is conserved following the motion. This "conservable" property of P is one reason why it is of such interest to dynamical meteorologists (see Sections 5.2.3 and 6.2.4, for example). A physical interpretation of the conservation of P is given in Section 3.8.2.

3.1.2 Boundary Conditions

To solve the primitive equations of Eqs. (3.1.3), or any approximate set of equations derived from them, it is of course necessary to apply suitable boundary conditions. These depend on the particular physical problem under consideration; some typical examples will be discussed here.

3.1.2.a Conditions at the Lower Boundary

1. If the lower boundary is the *ground*, the shape of the topography should be specified in terms of the geometric height z^*, rather than z (this is a slight inconvenience of log-pressure coordinates): for example,

$$z^* = h(x, y, t) \qquad \text{at the ground.}$$

(The t-dependence is a mathematical device that is sometimes useful for idealized initial-value problems; for example, a mountain might be "grown" so as to set up a flow in an unambiguous manner.) Since the ground is a material surface, the kinematic boundary condition is

$$\frac{D}{Dt}(z^* - h) = 0 \qquad \text{at} \quad z^* = h.$$

If viscosity is important, further conditions are required, but we shall not need them in this book. In terms of Φ, we have, from Eq. (1.1.3),

$$\frac{D\Phi}{Dt} = g\frac{Dh}{Dt} \qquad \text{at} \quad \Phi = \int_0^h g\, dz^* \approx gh, \tag{3.1.6a}$$

where the latter approximation relies on the fact that g is essentially constant over the altitude range of the earth's topography.

2. Some models specify the geopotential or geometric height of a given log-pressure level, for example,

$$\Phi(x, y, z_0, t) = F(x, y, t), \tag{3.1.6b}$$

where $z_0 = -H \ln(p_0/p_s) =$ constant; p_0 might be near the tropopause (say

$p_0 = 100$ mb). This is usually the easiest form of lower boundary condition to implement in log-pressure coordinates.

3. It is sometimes convenient in simple mathematical models to specify the log-pressure at a lower material boundary. Thus $z = \zeta(x, y, t)$, say, at the boundary; by analogy with (1), the kinematic boundary condition is then

$$w \equiv \frac{Dz}{Dt} = \frac{D\zeta}{Dt} \quad \text{at} \quad z = \zeta(x, y, t). \tag{3.1.6c}$$

This condition is less suitable than Eq. (3.1.6a) or (3.1.6b) for use in detailed simulation of the atmosphere.

3.1.2.b Conditions at the Upper Boundary

Numerical general circulation models of the middle atmosphere, such as those to be discussed in Chapter 11, employ a finite number of levels in the vertical and usually have to include an effectively "rigid" upper boundary. For example, a model formulated in z coordinates might take $w = 0$ at some large but finite height z_1. A rigid lid of this type will tend to lead to unrealistic reflections of wave disturbances that reach it, and large dissipative terms are usually introduced near the upper boundary in an attempt to damp such waves and minimize spurious reflections. These dissipation terms are primarily a numerical expedient and normally have little physical basis.

Simpler linear models of wave disturbances in the middle atmosphere can often adopt more satisfactory dynamical upper boundary conditions, which fall into one of two categories:

1. Disturbances are *trapped* or *evanescent*: that is, they tend to zero with increasing height (where a suitable measure of a disturbance might be its energy per unit volume).

2. Vertically propagating disturbances obey a *radiation condition*: that is, they transfer "information" upward, and not downward, at great heights. This condition tacitly assumes that mean atmospheric conditions do not allow significant reflection of vertically travelling disturbances at great heights, so that a clear distinction can be made between upward and downward propagation. The radiation condition then states that only the upward-propagating disturbances—as identified perhaps by a "group velocity" argument—exist above some height z_2.

We shall examine cases (1) and (2) in detail in Chapter 4, when specific examples of wave motions are discussed.

3.1.2.c Conditions at Side Boundaries

The conditions here depend on the geometry of the atmospheric model under consideration. On the sphere, it is only necessary that all variables be bounded at the poles. In idealized cases where attention is fixed on a "channel" with vertical walls parallel to latitude circles $\phi = \phi_1, \phi_2$, say, $v = 0$ is taken on these walls.

3.2. The Beta-Plane Approximation and Quasi-Geostrophic Theory

The primitive equations (3.1.3) are still a complicated set, despite the simplifications that have been used in deriving them. Moreover, they are capable of describing a very wide range of atmospheric flows, from slow motions of global scale to quite rapid, medium-scale disturbances. To focus on the larger-scale, slower motions, at least in the extratropical regions, we can introduce further approximations to obtain the *quasi-geostrophic equations.*

3.2.1 The Primitive Equations on a Beta-Plane

Before making the *dynamical* approximations that result in the quasi-geostrophic equations, it is first convenient to make a *geometrical* simplification, by replacing the spherical coordinates (λ, ϕ) by eastward and northward cartesian coordinates (x, y), and restricting the flow domain to some neighborhood of the latitude ϕ_0. This task can be carried out in a formally rigorous manner; however, since the resulting primitive equations are intuitively reasonable approximations to the full set of Eqs. (3.1.3), we shall only state them here. They are:

$$\frac{Du}{Dt} - fv + \Phi_x = X, \tag{3.2.1a}$$

$$\frac{Dv}{Dt} + fu + \Phi_y = Y, \tag{3.2.1b}$$

$$\Phi_z = H^{-1}R\theta e^{-\kappa z/H}, \tag{3.2.1c}$$

$$u_x + v_y + \rho_0^{-1}(\rho_0 w)_z = 0, \tag{3.2.1d}$$

$$\frac{D\theta}{Dt} = Q, \tag{3.2.1e}$$

with

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Here x is eastward distance and y northward distance from some origin (λ_0, ϕ_0), subscripts x and y denote partial derivatives, and other symbols are as before, with the exception that now

$$f = f_0 + \beta y, \tag{3.2.1f}$$

where $f_0 \equiv 2\Omega \sin \phi_0$ and $\beta \equiv 2\Omega a^{-1} \cos \phi_0$. Note that Eq. (3.2.1f) is a formally valid approximation to $f = 2\Omega \sin \phi$ if $|y| \ll a \cot \phi_0$; the terms in $\tan \phi$ in Eqs. (3.1.3a, b) are negligible if u and v are comparable in magnitude and their horizontal length scales are much less than $a \cot \phi_0$. The linear variation of f with y captures the most important dynamical effect of the variation of $2\Omega \sin \phi$ with latitude; this "beta-effect" was first pointed out by Rossby (1939). Equations (3.2.1) are called the "beta-plane" versions of the primitive equations. To simulate the periodicity around latitude circles on the sphere, it is often convenient to consider the beta-plane to be periodic in x with period $2\pi a \cos \phi_0$.

3.2.2 Geostrophic Balance and the Thermal Wind Equations

Having simplified the geometry in this way, we can next use the fact that for large-scale, low-frequency, extratropical flows, approximate *geostrophic balance* holds: that is, the Coriolis terms $(-fv, fu)$ in Eqs. (3.2.1a, b) are roughly balanced by the horizontal gradients of geopotential. Thus the horizontal wind $(u, v, 0)$ satisfies

$$u \approx u_g, \qquad v \approx v_g, \tag{3.2.2}$$

where the *geostrophic wind* $\mathbf{u}_g \equiv (u_g, v_g, 0)$ is defined in terms of the geopotential by

$$(u_g, v_g) \equiv (-\psi_y, \psi_x), \tag{3.2.3}$$

where

$$\psi \equiv f_0^{-1}(\Phi - \Phi_0) \tag{3.2.4}$$

is called the geostrophic stream function and $\Phi_0(z)$ is a suitable reference geopotential profile; note that the definition of ψ involves f_0 and not f. From the hydrostatic balance of Eq. (3.2.1c), we have

$$\theta_e \equiv \theta - \theta_0(z) = HR^{-1} f_0 e^{\kappa z/H} \psi_z, \tag{3.2.5}$$

where $\theta_0(z) = HR^{-1} e^{\kappa z/H} \Phi_{0z}$ is a reference potential temperature. Likewise, using Eq. (3.1.3c′),

$$T - T_0(z) = HR^{-1} f_0 \psi_z, \tag{3.2.5'}$$

where $T_0(z) = e^{-\kappa z/H} \theta_0(z)$ is a reference temperature. [Possible choices for $T_0(z)$ might be the midlatitude profile sketched in Fig. 1.1, or a global mean profile; $\Phi_0(z)$ would then be obtained by vertical integration, subject to the boundary condition $\Phi_0(0) = 0$.] Combining Eqs. (3.2.3) and (3.2.5) or (3.2.5′) to eliminate ψ by cross differentiation, we obtain the "thermal wind" equations

$$\frac{\partial u_g}{\partial z} = -\frac{R}{Hf_0} e^{-\kappa z/H} \frac{\partial \theta}{\partial y} = -\frac{R}{Hf_0} \frac{\partial T}{\partial y}, \tag{3.2.6a}$$

$$\frac{\partial v_g}{\partial z} = \frac{R}{Hf_0} e^{-\kappa z/H} \frac{\partial \theta}{\partial x} = \frac{R}{Hf_0} \frac{\partial T}{\partial x}, \tag{3.2.6b}$$

which relate the vertical shear of the geostrophic wind components to horizontal potential temperature (or temperature) gradients. Note also from Eq. (3.2.3) that $\partial u_g/\partial x + \partial v_g/\partial y \equiv 0$, so by the continuity equation [Eq. (3.2.1d)] the geostrophic wind is associated with a vertical "velocity" w_g that satisfies $(\rho_0 w_g)_z \equiv 0$. To ensure that w_g is bounded as $z \to \infty$, we must therefore take $w_g \equiv 0$.

3.2.3 Quasi-Geostrophic Flow

It will be observed that Eqs. (3.2.2)-(3.2.4) are first approximations to the horizontal momentum equations (3.2.1a, b), provided that the accelerations Du/Dt and Dv/Dt and the nonconservative terms X and Y are ignored, and $f_0 + \beta y$ is replaced by f_0. To examine these approximations more closely, and to investigate the time development of the geostrophic flow [which is not predicted by Eqs. (3.2.2)-(3.2.4)], we define ageostrophic velocities, denoted by a subscript a, thus:

$$u_a \equiv u - u_g, \qquad v_a \equiv v - v_g, \qquad w_a \equiv w. \tag{3.2.7}$$

We suppose that U is a typical order of magnitude of the geostrophic wind speed $|\mathbf{u}_g|$, and that L is a typical horizontal length scale, so that $\partial/\partial x$ and $\partial/\partial y$ are $O(L^{-1})$. It can then be shown that Eqs. (3.2.2)-(3.2.4) are valid first approximations, with $|u_a| \ll |u_g| \sim U$, $|v_a| \ll |v_g| \sim U$, if the following conditions are satisfied:

$$\begin{aligned} &\text{(a)} \qquad \mathrm{Ro} \equiv U/f_0 L \ll 1. \\ &\text{(b)} \qquad \partial/\partial t \ll f_0. \\ &\text{(c)} \qquad \beta L \ll f_0, \\ &\text{(d)} \qquad |X|, |Y| \ll f_0 U. \end{aligned} \tag{3.2.8}$$

Condition (a) states that the *Rossby number* Ro, which measures the ratio of the nonlinear terms $\mathbf{u} \cdot \nabla(u, v)$ to the Coriolis terms $(-f_0 v, f_0 u)$ in Eqs. (3.2.1a,b), should be small. Likewise, condition (b) states that the ratio of the time derivatives $(\partial u/\partial t,\ \partial v/\partial t)$ to the Coriolis terms should be small. Condition (c) allows the use of f_0 rather than f in Eq. (3.2.4), while condition (d) ensures that friction is small. These conditions make precise the restriction to "large-scale, low-frequency motions" mentioned above.

Given conditions (3.2.8), the next approximation to Eqs. (3.2.1a,b,d,e) beyond geostrophic balance is a set of equations describing "quasi-geostrophic flow":

$$D_g u_g - f_0 v_a - \beta y v_g = X, \tag{3.2.9a}$$

$$D_g v_g + f_0 u_a + \beta y u_g = Y, \tag{3.2.9b}$$

$$u_{ax} + v_{ay} + \rho_0^{-1}(\rho_0 w_a)_z = 0 \tag{3.2.9c}$$

$$D_g \theta_e + w_a \theta_{0z} = Q, \tag{3.2.9d}$$

where

$$D_g \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}$$

is the time derivative following the geostrophic wind. It is assumed that the departure θ_e from the reference temperature $\theta_0(z)$ is always small, in the sense that $|\theta_{ez}| \ll \theta_{0z}$, so that $w_a \theta_z$ can be replaced by $w_a \theta_{0z}$, as in Eq. (3.2.9d). This is a fair approximation in the middle atmosphere.

The quasi-geostrophic set of Eqs. (3.2.9) still appears quite complicated; however, we now combine the members to yield a single useful and illuminating equation, Eq. (3.2.14). First, we construct the *vorticity equation*

$$D_g \zeta_g = f_0 \rho_0^{-1}(\rho_0 w_a)_z - X_y + Y_x \tag{3.2.10}$$

by taking $(\partial/\partial x)$ [Eq. (3.2.9b)] $-\partial/\partial y$ [Eq. (3.2.9a)] and using the identities $\mathbf{u}_{gx} \cdot \nabla v_g \equiv 0$, $\mathbf{u}_{gy} \cdot \nabla u_g \equiv 0$, which follow from Eq. (3.2.3), and $D_g(f_0 + \beta y) \equiv \beta v_g$, together with Eq. (3.2.9c). Here

$$\zeta_g \equiv f_0 + \beta y - u_{gy} + v_{gx} = f_0 + \beta y + \psi_{xx} + \psi_{yy}$$

is the geostrophic approximation to the beta-plane form of the vertical component of the absolute vorticity, $f - u_y + v_x$. The first term on the right of Eq. (3.2.10) is called a "stretching" term, since it can generate vorticity by differential vertical motion.

The next step is to eliminate w_a between the thermodynamic equation [Eq. (3.2.9d)] and the vorticity equation [Eq. (3.2.10)]. We therefore multiply Eq. (3.2.9d) by ρ_0/θ_{0z}; this is a function of z alone, and can be taken through the D_g operator, giving

$$D_g(\rho_0 \theta_e/\theta_{0z}) + \rho_0 w_a = \rho_0 Q/\theta_{0z}. \tag{3.2.11}$$

Using Eq. (3.2.5), we can write

$$\theta_e/\theta_{0z} = f_0\psi_z/N^2, \tag{3.2.12}$$

where

$$N^2(z) \equiv H^{-1}R\theta_{0z}(z)e^{-\kappa z/H} \equiv \frac{R}{H}\left(T_{0z} + \frac{\kappa T_0}{H}\right). \tag{3.2.13}$$

The term N is thus the log-pressure buoyancy frequency corresponding to the reference temperature profile $T_0(z) \equiv \theta_0(z)e^{-\kappa z/H}$ [cf. Eq. (1.1.13)]. As indicated in Section 1.1.4, the atmosphere is statically stable if N^2 and θ_{0z} are positive.

On combining the z derivative of Eq. (3.2.11) with Eq. (3.2.10), using the identity $\mathbf{u}_{gz} \cdot \nabla(\rho_0\theta_e/\theta_{0z}) \equiv 0$ [which follows from Eq. (3.2.6) and the fact that ρ_0/θ_{0z} and θ_0 depend on z alone], and substituting Eq. (3.2.12), we obtain the *quasi-geostrophic potential vorticity equation*

$$D_g q_g = -X_y + Y_x + f_0\rho_0^{-1}(\rho_0 Q/\theta_{0z})_z, \tag{3.2.14}$$

where

$$q_g \equiv \zeta_g + f_0\rho_0^{-1}(\rho_0\theta_e/\theta_{0z})_z \tag{3.2.15a}$$

$$\equiv f_0 + \beta y + \psi_{xx} + \psi_{yy} + \rho_0^{-1}(\rho_0\varepsilon\psi_z)_z \tag{3.2.15b}$$

is the quasi-geostrophic potential vorticity and

$$\varepsilon(z) \equiv f_0^2/N^2(z). \tag{3.2.16}$$

Equation (3.2.14) gives the time development of q_g; note that it does not involve ageostrophic velocities. Moreover, if the flow is frictionless ($X = Y = 0$) and adiabatic ($Q = 0$), then $D_g q_g = 0$ and q_g is conserved following the geostrophic wind. Given q_g at any instant, and appropriate boundary conditions, the elliptic operator on the right of Eq. (3.2.15b) can in principle be inverted to obtain ψ, and hence u_g, v_g, and θ or T, using Eqs. (3.2.3) and (3.2.5).

The quasi-geostrophic potential vorticity q_g, defined by Eq. (3.2.15), should be contrasted with Ertel's potential vorticity P, defined by Eq. (3.1.4). In particular, q_g is *not* generally the quasi-geostrophic approximation to P. Furthermore, if $X = Y = Q = 0$, q_g is conserved following the horizontal geostrophic flow under the quasi-geostrophic conditions (3.2.8), while P is conserved following the total flow, even when quasi-geostrophic scaling is not valid: see Eq. (3.1.5). For these reasons, some authors call q_g the "pseudo-potential vorticity," although when there is no danger of confusion, we shall simply call q_g the "potential vorticity" for short. (Note, however, that certain analogies exist between formulas involving q_g in log-pressure coordinates and formulas involving P in isentropic coordinates: see Section 3.8.3.)

Another useful equation that can be derived from Eq. (3.2.9) is the *omega equation*, obtained by eliminating the $\partial/\partial t$ terms in Eqs. (3.2.9a,d). It is a *diagnostic* equation (that is, it involves no time derivatives) for obtaining the ageostrophic velocity w_a from the geostrophic quantity ψ and its derivatives. Some special cases of the omega equation will be discussed in Sections 3.3 and 3.5.

We note finally that several versions of the quasi-geostrophic equations have been derived in spherical coordinates. Although none of these is entirely satisfactory in every respect, some examples are useful in modeling the middle atmosphere and are mentioned in Chapters 5 and 6.

3.3 The Eulerian-Mean Equations

Many of the middle atmosphere phenomena to be discussed in this book can be regarded as involving the interaction of a mean flow with disturbances ("waves" or "eddies") that are superimposed upon it. This interaction is generally a two-way process, for the mean-flow configuration can strongly modify the propagation of the disturbances, while the disturbances themselves can bring about significant mean-flow changes, through rectified nonlinear effects.

We shall mostly be concerned with cases where the mean is a *zonal* mean, to be denoted by an overbar: thus, for example,

$$\bar{u}(\phi, z, t) = (2\pi)^{-1} \int_0^{2\pi} u(\lambda, \phi, z, t)\, d\lambda. \tag{3.3.1a}$$

The departure from the zonal mean will be denoted by a prime:

$$u'(\lambda, \phi, z, t) \equiv u - \bar{u}. \tag{3.3.1b}$$

It should be emphasized that this separation into mean and disturbance quantities is primarily a mathematical device and may not be the most natural physical separation in all cases; for example, in many tropospheric applications a *time* mean may be more useful. However, the zonal average has proved a satisfactory tool for the investigation of most of the stratospheric and mesospheric phenomena to be discussed in this book. Moreover, the theory of the interaction of waves with the zonal-mean flow is more highly developed than that for the corresponding interaction of transient disturbances with the time-mean flow.

The average defined by Eq. (3.3.1a) is an example of an *Eulerian* mean, since it is taken over λ at fixed values of the coordinates ϕ, z, and t. Another type of average, to be discussed in Section 3.7, is the Lagrangian mean, which is taken over a specified set of moving fluid parcels.

Separating each variable into a zonal-mean part and a disturbance part, as in Eq. (3.3.1b), substituting into Eq. (3.1.3), taking the zonal average, and performing some straightforward manipulations, we obtain the following set of primitive equations for the Eulerian-mean flow in spherical coordinates:

$$\bar{u}_t + \bar{v}[(a\cos\phi)^{-1}(\bar{u}\cos\phi)_\phi - f] + \bar{w}\bar{u}_z - \bar{X}$$
$$= -(a\cos^2\phi)^{-1}(\overline{v'u'}\cos^2\phi)_\phi - \rho_0^{-1}(\rho_0\overline{w'u'})_z, \tag{3.3.2a}$$

$$\bar{v}_t + a^{-1}\bar{v}\bar{v}_\phi + \bar{w}\bar{v}_z + \bar{u}(f + \bar{u}a^{-1}\tan\phi) + a^{-1}\bar{\Phi}_\phi - \bar{Y}$$
$$= -(a\cos\phi)^{-1}(\overline{v'^2}\cos\phi)_\phi - \rho_0^{-1}(\rho_0\overline{w'v'})_z - \overline{u'^2}a^{-1}\tan\phi, \tag{3.3.2b}$$

$$\bar{\Phi}_z - H^{-1}R\bar{\theta}e^{-\kappa z/H} = 0, \tag{3.3.2c}$$

$$(a\cos\phi)^{-1}(\bar{v}\cos\phi)_\phi + \rho_0^{-1}(\rho_0\bar{w})_z = 0, \tag{3.3.2d}$$

$$\bar{\theta}_t + a^{-1}\bar{v}\bar{\theta}_\phi + \bar{w}\bar{\theta}_z - \bar{Q}$$
$$= -(a\cos\phi)^{-1}(\overline{v'\theta'}\cos\phi)_\phi - \rho_0^{-1}(\rho_0\overline{w'\theta'})_z. \tag{3.3.2e}$$

(The subscript t denotes a time derivative.) In these equations the terms involving mean quadratic functions of disturbance variables have been written on the right. Given these "rectified eddy-forcing" terms, together with suitable expressions for $\bar{X}$, $\bar{Y}$, and $\bar{Q}$ and appropriate boundary and initial conditions, Eqs. (3.3.2) comprise a closed set of equations for predicting the time development of the zonal-mean circulation. A similar set of primitive equations can be written down for the Eulerian-mean flow on a beta-plane, starting from Eq. (3.2.1) and replacing Eq. (3.3.1a) by

$$\bar{u}(y, z, t) \equiv a_0^{-1}\int_0^{a_0} u(x, y, z, t)\, dx,$$

where $a_0 \equiv 2\pi a\cos\phi_0$ is the length of the latitude circle at $\phi = \phi_0$.

In the case of quasi-geostrophic flow on a beta-plane we first note that

$$\bar{v}_g \equiv a_0^{-1}\int_0^{a_0} \psi_x\, dx = 0,$$

by periodicity, so the zonal-mean geostrophic wind $(\bar{u}_g, \bar{v}_g, 0)$ is purely zonal. Dropping subscripts g on geostrophic quantities, but retaining subscripts a on ageostrophic variables, we then obtain the set

$$\bar{u}_t - f_0\bar{v}_a - \bar{X} = -(\overline{v'u'})_y, \tag{3.3.3a}$$

$$\bar{\theta}_t + \bar{w}_a\theta_{0z} - \bar{Q} = -(\overline{v'\theta'})_y, \tag{3.3.3b}$$

$$\bar{v}_{ay} + \rho_0^{-1}(\rho_0\bar{w}_a)_z = 0, \tag{3.3.3c}$$

$$f_0\bar{u}_z + H^{-1}Re^{-\kappa z/H}\bar{\theta}_y = 0, \tag{3.3.3d}$$

from Eqs. (3.2.9a,c,d) and (3.2.6a), after a little manipulation. These again form a closed set for the mean-flow variables ($\bar{u}$, $\bar{\theta}$, $\bar{v}_a$, $\bar{w}_a$), given the rectified eddy-forcing terms on the right, $\bar{X}$, $\bar{Q}$, and suitable boundary conditions. [The zonal mean of the y-momentum equation, Eq. (3.2.9b), then supplies $\bar{u}_a$, if $\bar{Y}$ and further rectified eddy terms are given.] From Eqs. (3.2.14) and (3.2.3) it is easy to obtain the zonal-mean quasi-geostrophic potential vorticity equation

$$\bar{q}_t + (\overline{v'q'})_y = -\bar{X}_y + f_0\rho_0^{-1}(\rho_0\bar{Q}/\theta_{0z})_z. \tag{3.3.4}$$

The y derivative of this equation can also be obtained by elimination of ($\bar{\theta}_t$, $\bar{v}_a$, $\bar{w}_a$) from Eqs. (3.3.3), using Eqs. (3.2.3) and (3.2.15), provided that Eq. (3.5.10) is used to relate the mean northward eddy potential vorticity flux $\overline{v'q'}$ to $\overline{v'u'}$ and $\overline{v'\theta'}$; see the end of Section 3.5. One can also eliminate $\bar{u}_t$ and $\bar{\theta}_t$ from Eqs. (3.3.3) to obtain diagnostic equations for $\bar{v}_a$ and $\bar{w}_a$ analogous to the omega equation mentioned above. A similar equation, Eq. (3.5.8), will be discussed.

3.4 Linearized Disturbances to Zonal-Mean Flows

In the preceding section we briefly discussed the separation of atmospheric flows into Eulerian zonal-mean and "wave" or "eddy" parts, and presented sets of equations governing the zonal-mean flow. Similar sets of equations also hold for the disturbances to the zonal mean: exact forms of such equations are given for example by Holton (1975), Eqs. (2.19)-(2.22). In practice, these disturbance equations are most useful in studies of *small-amplitude* departures from the zonal-mean state, when they can be linearized in the disturbance amplitude and perhaps solved numerically, or even analytically in simple idealized cases. A variety of solutions of this type will be described later in this book, and we shall present here the appropriate sets of linearized equations, for future reference.

We first consider a steady, zonally symmetric basic flow, which is purely zonal and unforced. Denoting the basic state by an overbar and suffix zero, we thus have $\bar{v}_0 = \bar{w}_0 = 0$, and, from Eq. (3.1.3b,c),

$$\left(f + \frac{\bar{u}_0 \tan\phi}{a}\right)\bar{u}_0 + a^{-1}\bar{\Phi}_{0\phi} = 0, \tag{3.4.1a}$$

$$\bar{\Phi}_{0z} = H^{-1}R\bar{\theta}_0 e^{-\kappa z/H}, \tag{3.4.1b}$$

in the spherical, primitive equation case.[1] [The presence of a basic diabatic

[1] Note that $\bar{\theta}_0(\phi, z)$, $\bar{T}_0(\phi, z)$, and $\bar{\Phi}_0(\phi, z)$ are not in general equal to the reference profiles $\theta_0(z)$, $T_0(z)$, and $\Phi_0(z)$ introduced in Section 3.2.2.

heating $\bar{Q}_0$ would be associated with nonzero $(\bar{v}_0, \bar{w}_0)$ by Eq. (3.1.3e), and would introduce extra complications that will not concern us here.] Note that elimination of $\bar{\Phi}_0$ from Eqs. (3.4.1a,b) gives the thermal wind equation for the basic state [cf. Eq. (3.2.6a)]:

$$\left(f + \frac{2\bar{u}_0 \tan\phi}{a}\right)\frac{\partial \bar{u}_0}{\partial z} = \frac{-R}{aH}\frac{\partial \bar{\theta}_0}{\partial \phi} e^{-\kappa z/H} = \frac{-R}{aH}\frac{\partial \bar{T}_0}{\partial \phi}. \tag{3.4.1c}$$

We now consider small disturbances to this basic state. Thus all primed quantities, as defined in Section 3.3, will be taken to be $O(\alpha)$, where α is a dimensionless amplitude parameter that is much less than 1. Furthermore, $\bar{u} - \bar{u}_0$, $\bar{\Phi} - \bar{\Phi}_0$, $\bar{\theta} - \bar{\theta}_0$, $\bar{v}$, $\bar{w}$, $\bar{X}$, $\bar{Y}$ and $\bar{Q}$ must all be $O(\alpha^2)$ or smaller. The self-consistency of these conditions on the disturbances and the mean flow follows from Eqs. (3.3.2) and (3.4.1); for example, the eddy-forcing terms on the right of Eqs. (3.3.2a,b,e) can lead to $O(\alpha^2)$ departures of $\bar{u}$ from $\bar{u}_0$, and $X = O(\alpha^2)$ would do likewise.

Substitution of

$$u = \bar{u}_0 + u' + O(\alpha^2), \qquad v = v' + O(\alpha^2), \qquad \text{etc.}$$

into the primitive equations of Eq. (3.1.3) and use of Eq. (3.4.1) then give the following set of linear equations for the disturbances:

$$\bar{D}u' + [(a\cos\phi)^{-1}(\bar{u}\cos\phi)_\phi - f]v' + \bar{u}_z w' + (a\cos\phi)^{-1}\Phi'_\lambda = X', \tag{3.4.2a}$$

$$\bar{D}v' + (f + 2\bar{u}a^{-1}\tan\phi)u' + a^{-1}\Phi'_\phi = Y', \tag{3.4.2b}$$

$$\Phi'_z = H^{-1}R\theta' e^{-\kappa z/H}, \tag{3.4.2c}$$

$$(a\cos\phi)^{-1}[u'_\lambda + (v'\cos\phi)_\phi] + \rho_0^{-1}(\rho_0 w')_z = 0, \tag{3.4.2d}$$

$$\bar{D}\theta' + a^{-1}\bar{\theta}_\phi v' + \bar{\theta}_z w' = Q'. \tag{3.4.2e}$$

Here terms of $O(\alpha^2)$ have been neglected and (consistent with this approximation) $\bar{u}_0$ and $\bar{\theta}_0$ have been replaced by $\bar{u}$ and $\bar{\theta}$, respectively, to simplify the notation. Moreover,

$$\bar{D} \equiv \frac{\partial}{\partial t} + \frac{\bar{u}}{a\cos\phi}\frac{\partial}{\partial \lambda} \tag{3.4.3}$$

is the time derivative following the basic flow. The corresponding beta-plane versions can be obtained in a similar manner from Eqs. (3.2.1).

An analogous linearization procedure can be performed for the full quasi-geostrophic set of Eq. (3.2.9). We just note here the equations for a steady basic geostrophic zonal flow, which follow from Eqs. (3.2.3)–(3.2.5):

$$\bar{u} = -\bar{\psi}_y = -f_0^{-1}\bar{\Phi}_y, \tag{3.4.4a}$$

$$\bar{\theta} - \theta_0(z) = HR^{-1}f_0 e^{\kappa z/H}\bar{\psi}_z, \tag{3.4.4b}$$

and the linearized version of the quasi-geostrophic potential vorticity equation, Eq. (3.2.14):

$$\bar{D}q' + v'\bar{q}_y = -X'_y + Y'_x + f_0\rho_0^{-1}(\rho_0 Q'/\theta_{0z})_z. \tag{3.4.5}$$

Here

$$\bar{D} \equiv \frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} \tag{3.4.6}$$

is the time derivative following the basic flow,

$$q' \equiv \psi'_{xx} + \psi'_{yy} + \rho_0^{-1}(\rho_0 \varepsilon \psi'_z)_z \tag{3.4.7}$$

is the disturbance potential vorticity, and

$$\bar{q}_y \equiv \beta - \bar{u}_{yy} - \rho_0^{-1}(\rho_0 \varepsilon \bar{u}_z)_z \tag{3.4.8}$$

is the basic northward potential vorticity gradient (sometimes called "effective beta"). The subscript g on geostrophic quantities has again been omitted here, as has the subscript zero on $\bar{u}$, etc. Note that Eq. (3.4.8) follows from the y derivative of the zonal mean of Eq. (3.2.15), together with Eq. (3.4.4a). Versions of Eq. (3.4.5) in spherical coordinates can also be written down; an example is Eq. (5.3.1).

The technique of expansion in the small-amplitude parameter α, described here, can be carried to higher orders. For example, on using $O(\alpha)$ wave solutions of Eq. (3.4.2) to calculate the rectified eddy terms on the right of the Eulerian-mean equations, Eqs. (3.3.2) [or, more conveniently, of the transformed set of Eqs. (3.5.2)], one can in principle calculate the $O(\alpha^2)$ back-effect of the waves on the mean flow. Examples of this method will be discussed in Sections 6.3.1 and 8.3.2. Of course, such an asymptotic expansion in amplitude can only describe weakly nonlinear aspects of the interaction of waves and mean flows. Apart from some rather exceptional circumstances under which exact analytical solutions for finite-amplitude disturbances can be constructed, the behavior of large-amplitude waves must be investigated by numerical solution of the full nonlinear equations.

3.5 The Transformed Eulerian-Mean Equations

The Eulerian-mean sets of equations presented in Section 3.3 were obtained by a straightforward separation of the atmospheric variables into mean and disturbance parts, and averaging and manipulation of the basic equations. However, it is less easy to anticipate how the zonal-mean flow will respond, for example, to a specified "eddy momentum flux" $\overline{v'u'}$ or "eddy heat flux" $\overline{v'\theta'}$ in the quasi-geostrophic set of Eqs. (3.3.3), or in turn to anticipate what physical properties of the waves control these eddy fluxes. To investigate questions like these, it is convenient to transform the mean-flow equations to an alternative form. In the spherical case, the approach is first to define a *residual mean meridional circulation* $(0, \bar{v}^*, \bar{w}^*)$ by

$$\bar{v}^* \equiv \bar{v} - \rho_0^{-1}(\rho_0 \overline{v'\theta'}/\bar{\theta}_z)_z, \tag{3.5.1a}$$

$$\bar{w}^* \equiv \bar{w} + (a \cos \phi)^{-1}(\cos \phi\, \overline{v'\theta'}/\bar{\theta}_z)_\phi. \tag{3.5.1b}$$

(Other definitions of the residual circulation are also possible.) On substituting for $(\bar{v}, \bar{w})$ in Eqs. (3.3.2), using Eqs. (3.5.1) and rearranging, the following *transformed Eulerian-mean* (TEM) set is obtained:

$$\bar{u}_t + \bar{v}^*[(a \cos \phi)^{-1}(\bar{u} \cos \phi)_\phi - f] + \bar{w}^*\bar{u}_z - \bar{X}$$
$$= (\rho_0 a \cos \phi)^{-1}\nabla \cdot \mathbf{F}, \tag{3.5.2a}$$

$$\bar{u}(f + \bar{u}a^{-1} \tan \phi) + a^{-1}\bar{\Phi}_\phi = G, \tag{3.5.2b}$$

$$\bar{\Phi}_z - H^{-1}R\bar{\theta}e^{-\kappa z/H} = 0, \tag{3.5.2c}$$

$$(a \cos \phi)^{-1}(\bar{v}^* \cos \phi)_\phi + \rho_0^{-1}(\rho_0 \bar{w}^*)_z = 0, \tag{3.5.2d}$$

$$\bar{\theta}_t + a^{-1}\bar{v}^*\bar{\theta}_\phi + \bar{w}^*\bar{\theta}_z - \bar{Q} = -\rho_0^{-1}[\rho_0(\overline{v'\theta'}\, \bar{\theta}_\phi/a\bar{\theta}_z + \overline{w'\theta'})]_z. \tag{3.5.2e}$$

The vector $\boldsymbol{F} \equiv (0, F^{(\phi)}, F^{(z)})$ is known as the *Eliassen-Palm flux* (EP flux); its components are given by

$$F^{(\phi)} \equiv \rho_0 a \cos \phi(\bar{u}_z \overline{v'\theta'}/\bar{\theta}_z - \overline{v'u'}), \tag{3.5.3a}$$

$$F^{(z)} \equiv \rho_0 a \cos \phi\{[f - (a \cos \phi)^{-1}(\bar{u} \cos \phi)_\phi]\overline{v'\theta'}/\theta_z - \overline{w'u'}\}; \tag{3.5.3b}$$

note that

$$\nabla \cdot \mathbf{F} \equiv (a \cos \phi)^{-1} \frac{\partial}{\partial \phi}(F^{(\phi)} \cos \phi) + \frac{\partial F^{(z)}}{\partial z}$$

in spherical, log-pressure coordinates. In Eq. (3.5.2b) G represents all the terms that lead to a departure from gradient-wind balance between $\bar{u}$ and $\bar{\Phi}$; it can readily be calculated from Eqs. (3.3.2b) and (3.5.1). In most meteorological applications G is small and only produces slight deviations from gradient wind balance; its dynamical effects are usually only of secondary importance.

At first sight, the transformed set of Eqs. (3.5.2) appears to have no particular advantage over the Eulerian-mean equations of Eqs. (3.3.2). However, a more detailed investigation, to be described in Section 3.6, shows that the rectified eddy-forcing terms on the right of Eqs. (3.5.2a,e) depend on certain basic physical properties of the wave or eddy disturbances. For example, Eq. (3.6.1) will show that $\nabla \cdot \mathbf{F} \equiv 0$ if the disturbances are steady, linear, frictionless, and adiabatic and if the mean flow is conservative to $O(\alpha)$; a similar result holds for the expression on the right of Eq. (3.5.2e). By contrast, under the same linear, steady, conservative conditions, the forcings on the right of Eqs. (3.3.2a,e) are nonzero in general. Further discussion of these results, and some important consequences, will be presented in the next section.

Similar sets of transformed equations can be derived for beta-plane geometry. We shall just discuss the quasi-geostrophic case, for which a residual circulation can be defined by

$$\bar{v}^* \equiv \bar{v}_a - \rho_0^{-1}(\rho_0 \overline{v'\theta'}/\theta_{0z})_z, \qquad \bar{w}^* \equiv \bar{w}_a + (\overline{v'\theta'}/\theta_{0z})_y; \tag{3.5.4}$$

from Eqs. (3.3.3) the following quasi-geostrophic TEM set can readily be obtained:

$$\bar{u}_t - f_0 \bar{v}^* - \bar{X} = \rho_0^{-1}\nabla \cdot \mathbf{F}, \tag{3.5.5a}$$

$$\bar{\theta}_t + \bar{w}^* \theta_{0z} - \bar{Q} = 0, \tag{3.5.5b}$$

$$\bar{v}^*_y + \rho_0^{-1}(\rho_0 \bar{w}^*)_z = 0, \tag{3.5.5c}$$

$$f_0 \bar{u}_z + H^{-1} R e^{-\kappa z/H} \bar{\theta}_y = 0. \tag{3.5.5d}$$

In this quasi-geostrophic beta-plane case,

$$\mathbf{F} \equiv (0, -\rho_0 \overline{v'u'}, \rho_0 f_0 \overline{v'\theta'}/\theta_{0z}). \tag{3.5.6}$$

Note that the only explicit appearance of eddy-forcing terms here is in $\rho_0^{-1}\nabla \cdot \mathbf{F}$ in the transformed mean zonal momentum equation, Eq. (3.5.5a); in particular, the eddy forcing of the quasi-geostrophic transformed mean thermodynamic equation, Eq. (3.5.5b), is negligible. Thus, as far as their effects on the mean tendencies $\bar{u}_t$ and $\bar{\theta}_t$ and on the residual circulation $(\bar{v}^*, \bar{w}^*)$ are concerned, the eddy momentum flux $\overline{v'u'}$ and eddy heat flux $\overline{v'\theta'}$ do not act separately [as might have been expected from the untransformed Eqs. (3.3.3)] but in the combination

$$\nabla \cdot \mathbf{F} \equiv -(\rho_0 \overline{v'u'})_y + (\rho_0 f_0 \overline{v'\theta'}/\theta_{0z})_z.$$

This latter point can be emphasized by solving Eqs. (3.5.5) [or Eqs. (3.3.3)] to find the mean tendencies and the residual circulation. For example, it can be shown that

$$\rho_0\left[\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \varepsilon \frac{\partial}{\partial z}\right)\right]\bar{u}_t = (\nabla \cdot \mathbf{F} + \rho_0 \bar{X})_{yy} - (\rho_0 f_0 \bar{Q}/\theta_{0z})_{yz} \tag{3.5.7}$$

and

$$\rho_0\left[\frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \varepsilon \frac{\partial}{\partial z}\right)\right] f_0 \bar{v}^*$$
$$= -[\rho_0 \varepsilon(\rho_0^{-1}\nabla \cdot \mathbf{F} + \bar{X})_z]_z - (\rho_0 f_0 \bar{Q}/\theta_{0z})_{yz}. \tag{3.5.8}$$

(The second of these is related to the omega equation.) Here the rectified effects of the eddies are expressed by the terms in $\nabla \cdot \mathbf{F}$ on the right, while friction and diabatic heating are given by $\bar{X}$ and $\bar{Q}$, respectively. Note that these forcing terms generally produce nonlocal responses in $\bar{u}_t$ and $\bar{v}^*$, since the operator in square brackets on the left of Eqs. (3.5.7) and (3.5.8) is elliptic. To solve for $\bar{u}_t$, $\bar{v}^*$, etc., given the right-hand sides, boundary conditions must be imposed. (See Appendix 3B.)

We conclude this section by mentioning a useful alternative form for $\rho_0^{-1}\nabla \cdot \mathbf{F}$, valid under quasi-geostrophic scaling. It can be derived by simple manipulations using the following identities, which stem from Eqs. (3.2.3), (3.2.12), and (3.4.7):

$$u' = -\psi'_y, \qquad v' = \psi'_x, \qquad \theta'/\theta_{0z} = f_0\psi'_z/N^2,$$
$$q' = \psi'_{xx} + \psi'_{yy} + \rho_0^{-1}(\rho_0 \varepsilon \psi'_z)_z. \tag{3.5.9}$$

Some integrations by parts, and use of the fact that the x derivatives of zonal-mean quantities vanish, then yield

$$\begin{aligned}\overline{v'q'} &= -(\overline{v'u'})_y + \rho_0^{-1}(\rho_0 f_0 \overline{v'\theta'}/\theta_{0z})_z,\\ &= \rho_0^{-1}\nabla \cdot \mathbf{F}.\end{aligned} \tag{3.5.10}$$

This important quasi-geostrophic relationship between the northward eddy flux of potential vorticity and the divergence of the Eliassen–Palm flux can be used, together with Eq. (3.4.8), to show that Eq. (3.5.7) is equivalent to the y derivative of the quasi-geostrophic potential vorticity equation, Eq. (3.3.4): see the end of Section 3.3.

3.6 The Generalized Eliassen–Palm Theorem and the Charney–Drazin Nonacceleration Theorem

It was mentioned in the previous section that the divergence of the EP flux $\nabla \cdot \mathbf{F}$, unlike the convergence of the eddy momentum flux in Eq. (3.3.2a), depends on certain basic physical properties of the flow. This was given as the main reason for using the TEM set of Eqs. (3.5.2) in preference to the Eulerian-mean equations, Eqs. (3.3.2); we now discuss this point in more detail.

The foundations for the theory to be described were laid in a pioneering paper by Eliassen and Palm (1961). They considered steady, linear waves on a basic zonal flow $\bar{u}(\phi, z)$, with no frictional or diabatic effects ($X = Y = Q = 0$). Using a set of linear disturbance equations essentially equivalent to Eqs. (3.4.2) (but in pressure, rather than log-pressure, coordinates and beta-plane, rather than spherical, geometry), with $\partial/\partial t = 0$ and $X' = Y' = Q' = 0$, they proved the identity

$$\nabla \cdot \mathbf{F} \equiv 0. \tag{3.6.1}$$

Thus the divergence of the Eliassen-Palm flux (as it has come to be called) vanishes for linear, steady, conservative waves on a purely zonal basic flow. This result was extended to include nonzero X', Y', Q' and to allow for spherical geometry by Boyd (1976), and to include time-varying wave amplitudes (i.e., "wave transience") as well by Andrews and McIntyre (1976a, 1978a). The latter's *generalized Eliassen-Palm theorem* takes the form

$$\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = D + O(\alpha^3), \tag{3.6.2}$$

where A and D, like $\mathbf{F}$, are mean quadratic functions of disturbance quantities; however, unlike $\mathbf{F}$, their explicit primitive-equation forms generally involve parcel displacements and are quite complicated. [The simpler quasi-geostrophic versions are given in Eqs. (3.6.6), (3.6.7), and (3.6.10).] The "density" A appearing in Eq. (3.6.2) is called the "wave-activity density"; its time derivative represents wave-transience effects, vanishing for steady waves. The quantity D contains the frictional and diabatic effects X', Y', and Q', and thus vanishes for conservative waves. The $O(\alpha^3)$ term, where α is the wave amplitude as before, represents nonlinear wave effects, and vanishes for purely linear waves. [Note that Eq. (3.6.2), like the linearized Eqs. (3.4.2) from which it is derived, requires that $\bar{X}$, $\bar{Y}$, and $\bar{Q}$ are no larger than $O(\alpha^2)$, so that the basic flow is essentially zonal.]

The generalized EP theorem [Eq. (3.6.2)] makes explicit the dependence of $\nabla \cdot \mathbf{F}$ on the physical properties of wave transience and nonconservative wave effects. (Investigation of its detailed dependence on nonlinear wave processes generally involves going to higher orders in α.) More fundamentally, when the terms on the right of Eq. (3.6.2) are zero, it takes the form of a *conservation law* for wave properties: such laws are of considerable interest in many branches of physics. Note that it is simpler in structure than the *wave-energy equation*, which takes the form

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho_0\left(\overline{u'^2} + \overline{v'^2} + \overline{\Phi_z'^2}/N^2\right)\right] + \nabla \cdot \left(0, \rho_0\overline{v'\Phi'}, \rho_0\overline{w'\Phi'}\right)$$
$$= -\rho_0 a^{-1}[\bar{u}_\phi + \bar{u}\tan\phi]\overline{v'u'} - \rho_0\bar{u}_z\overline{w'u'} - \rho_0 a^{-1}\bar{\theta}_\phi\overline{v'\Phi_z'}/\bar{\theta}_z \tag{3.6.3}$$

for linear, conservative waves. This equation can be derived by taking $u' \times$ Eq. (3.4.2a) + $v' \times$ Eq. (3.4.2b) + $(\Phi_z'/\bar{\theta}_z) \times$ Eq. (3.4.2e), averaging, setting $X' = Y' = Q' = 0$, and using Eqs. (3.4.2c,d) and (3.2.13). The terms on the right of Eq. (3.6.3) are generally nonzero in the presence of a mean shear flow and represent an exchange of energy between the mean flow and the disturbances: no such exchange terms appear in the generalized EP theorem [Eq. (3.6.2)]. (In this respect the latter theorem is similar to the law of conservation of wave action: see Section 3.7.1 and Appendix 4A.) For these reasons Eq. (3.6.2) has certain advantages as a diagnostic of wave propagation in complicated mean flows, and will be used for such a purpose later in this book: see, for example, Sections 4.5.5, 5.2.2, 6.2.3, and 6.3.2.

We can now return to the TEM equations [Eqs. (3.5.2)], and use Eq. (3.6.2) to substitute for $\nabla \cdot \mathbf{F}$ on the right of Eq. (3.5.2a). By Eq. (3.6.1) that term vanishes if the disturbances are steady and linear, and the flow is conservative; a similar result can be shown to hold for the expression on the right of Eq. (3.5.2e). It then follows that under such hypotheses, and with appropriate boundary conditions (Appendix 3B), a possible mean flow satisfying Eq. (3.5.2) is given by

$$\bar{u}_t = \bar{\theta}_t = \bar{v}^* = \bar{w}^* = 0.$$

This is an example of a *nonacceleration theorem*, of which a first case was noted by Charney and Drazin (1961). It shows how the waves induce no mean-flow changes under the stated conditions; such a result is not at all obvious from the untransformed Eulerian-mean equations, for which the eddy-forcing terms on the right of Eqs. (3.3.2a,e) do not generally vanish when $\nabla \cdot \mathbf{F} = 0$, but induce a nonzero Eulerian-mean circulation $(\bar{v}, \bar{w})$ that precisely cancels their effect. [Note incidentally that zonally symmetric oscillations, involving a significant contribution $-\partial \bar{v}^*/\partial t$ to G in Eq. (3.5.2b), are possible in principle; however, these are not forced by the waves.]

As a result of the theory described above, there has been a recent emphasis on the physical processes that violate the nonacceleration theorem. Examples of such processes in wave, mean-flow interaction phenomena in the middle atmosphere will occur several times in this book.

We note finally the explicit quasi-geostrophic form of the generalized EP theorem. This can be derived most readily from the linearized potential vorticity equation, Eq. (3.4.5), which can be written

$$\bar{D}q' + v'\bar{q}_y = Z' + O(\alpha^2), \tag{3.6.4}$$

where $Z' \equiv -X'_y + Y'_x + f_0\rho_0^{-1}(\rho_0 Q'/\theta_{0z})_z$ and the $O(\alpha^2)$ term is the error incurred by linearization. On multiplying by $\rho_0 q'/\bar{q}_y$, taking $\rho_0(\bar{q}_y)^{-1}$ through the $\bar{D}$ operator with $O(\alpha^3)$ error [since $\bar{q}_t = O(\alpha^2)$ by Eq. (3.3.4) under the present hypotheses that $\bar{X} = \bar{Q} = O(\alpha^2)$], and averaging, we obtain

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_0\overline{q'^2}/\bar{q}_y) + \nabla \cdot \mathbf{F} = \rho_0\overline{Z'q'}/\bar{q}_y + O(\alpha^3), \tag{3.6.5}$$

using Eq. (3.5.10). This is of the form of Eq. (3.6.2) with quasi-geostrophic wave-activity density

$$A = \tfrac{1}{2}\rho_0 \overline{q'^2}/\bar{q}_y, \tag{3.6.6}$$

proportional to the "wave potential enstrophy," $\frac{1}{2}\overline{q'^2}$, and nonconservative term

$$D = \rho_0 \overline{Z'q'}/\bar{q}_y. \tag{3.6.7}$$

Note that A is positive definite if $\bar{q}_y > 0$ and is then a natural measure of wave amplitude. A useful alternative form for A is in terms of the northward parcel displacement η', defined by

$$\bar{D}\eta' = v' + O(\alpha^2). \tag{3.6.8}$$

From Eqs. (3.6.4) and (3.6.8) it follows that if $Z' = 0$,

$$q' = -\eta'\bar{q}_y + O(\alpha^2), \tag{3.6.9}$$

given suitable initial conditions, say $\eta' = q' = 0$ at $t = 0$. From Eqs. (3.6.6) and (3.6.9),

$$A = \tfrac{1}{2}\rho_0 \overline{\eta'^2}\bar{q}_y \tag{3.6.10}$$

if $Z' = 0$. When $Z' \neq 0$ we can retain Eq. (3.6.10) as the wave-activity density, but the corresponding nonconservative term D differs from that given in Eq. (3.6.7).

3.7 The Lagrangian Approach

3.7.1 Finite-Amplitude Theory

The generalized EP theorem and its corollary, the nonacceleration theorem, were derived in the previous section for disturbances of small amplitude α. However, many wave-like phenomena in the middle atmosphere are of large amplitude, and it is natural to inquire whether similar results apply to such waves. As yet, the only finite-amplitude results using the formalism adopted above have been rather restricted in character, although a promising approach has been developed by Killworth and McIntyre (1985).

Further progress along these lines has come from a rather different procedure, using a generalized Lagrangian-mean (GLM) theory, rather than the Eulerian-mean formalism discussed above. The GLM approach is quite technical in nature, and only a brief descriptive outline will be given here.

As its name implies, the GLM formalism involves taking averages following fluid parcels, rather than averaging over a set of coordinates fixed in

(λ, ϕ, z, t)-space, as with the Eulerian mean [Eq. (3.3.1a)]. The simplest Lagrangian mean to visualize is a time average following a single parcel; that this can differ significantly from the Eulerian mean is demonstrated by the trajectory in Fig. 3.1, which traces the motion of a single parcel in a hypothetical oscillatory flow whose Eulerian time mean is zero. By contrast, as shown by the mean drift of the parcel towards increasing x and y, the time-mean velocity following the parcel is nonzero for this flow.

For many meteorological purposes, however, a Lagrangian zonal average is required, and this can be described as follows. Consider an initially undisturbed, purely zonal basic flow (on a beta-plane, for simplicity), and fix attention on a thin, infinitely long tube of fluid, lying along the x axis (Fig. 3.2a). Suppose that some waves are excited: the tube will distort in a wavy manner (Fig. 3.2b) and its *mean* motion in the yz plane can be defined as the motion of a line R, which is parallel to the x axis and passes through the tube's center of mass as viewed in the yz plane. This construction gives the y and z components $(\bar{v}^{\mathrm{L}}, \bar{w}^{\mathrm{L}})$ of the Lagrangian-mean motion at the current position of R. A more general approach, associating each fluid parcel (P_{T}, say) in the tube with a suitably defined reference point P_{R} on R, allows the definition of a parcel displacement vector $\boldsymbol{\xi} = P_{\mathrm{R}}P_{\mathrm{T}}$, and also enables the Lagrangian means of other variables $(\bar{u}^{\mathrm{L}}, \bar{\theta}^{\mathrm{L}}$, etc.) to be defined.

A mathematical theory can be constructed using ideas like these; it provides in principle an exact finite-amplitude conservation law of the form

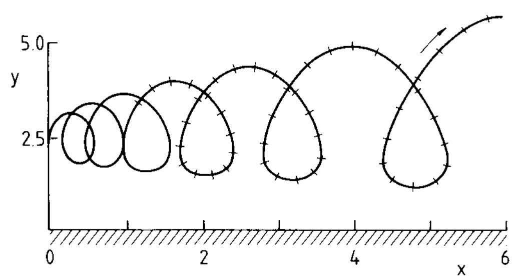

Fig. 3.1. Path of a single fluid parcel in the two-dimensional velocity field $u = 0.01t\cos(x - t)$, $v = 0.01yt\sin(x - t)$, satisfying the incompressibility condition $u_x + v_y = 0$. The path $[X(t), Y(t)]$ was calculated numerically by solving the equations $dX/dt = u(X, t)$, $dY/dt = v(X, Y, t)$, starting at the initial position $X = 0$, $Y = 2.5$. Part of the path is marked at equal time intervals $\Delta t = 0.5$. In addition to the clear mean drift to the right, the marks show that the parcel spends more time further from the wall as t increases so that its time-mean position (averaged over several cycles) also drifts away from the wall. On the other hand, a parcel starting on the wall $y = 0$ must remain on the wall since $v = 0$ there; the Lagrangian-mean motion is therefore divergent, even though (u, v) is nondivergent and the Eulerian-mean motion is zero. [After McIntyre (1980b).]

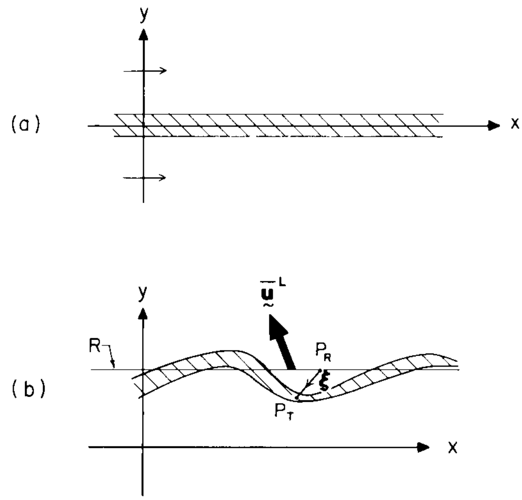

Fig. 3.2. Schematic illustration of the definitions of the Lagrangian-mean velocity $\bar{\mathbf{u}}^L$ and the parcel displacement $\boldsymbol{\xi}$. The material tube is shown hatched. See text for details.

of Eq. (3.6.2) (but with no error term), which generalizes the *wave-action* law of Bretherton and Garrett (1968). The theory also gives a set of equations for the Lagrangian-mean flow, which leads to a finite-amplitude nonacceleration theorem. These equations bear some formal similarity to the TEM set of Eqs. (3.5.2); however, it should be emphasized that the residual circulation $(0, \bar{v}^*, \bar{w}^*)$ is generally *not* the same as the Lagrangian mean meridional circulation $(0, \bar{v}^L, \bar{w}^L)$. These two velocities differ by terms representing transience, nonconservative effects, and nonlinearity in the waves. An important consequence of this difference is that, while the residual circulation has zero mass flux divergence, that is, $\nabla \cdot (0, \rho_0 \bar{v}^*, \rho_0 \bar{w}^*) = 0$ [see Eq. (3.5.2d)], the Lagrangian-mean flow is generally divergent, owing to the dispersion of parcels about their reference positions when waves are transient: see Eq. (9.4.16). (The same effect is also evident in the example shown in Fig. 3.1.)

Direct application of finite-amplitude GLM theory to atmospheric data, or even to numerical models of the atmosphere, encounters serious practical difficulties (see Section 6.3.2), although the theory has an obvious conceptual value for discussion of the transport of quasi-conservative tracers (see Section 9.4.2). A modified version of the theory, based on the use of the quasi-conservative tracers θ and P, may perhaps turn out to be of more practical meteorological benefit.

3.7.2 Small-Amplitude Theory

In the case of small-amplitude disturbances to a basic zonal flow $[\bar{u}(y, z), 0, 0]$, explicit but approximate calculations can be made of the Lagrangian quantities mentioned above, and these can provide helpful insights into atmospheric behavior. We shall illustrate this approach, using beta-plane geometry for simplicity.

First, the Cartesian components (ξ', η', ζ') of the parcel displacement vector $\boldsymbol{\xi}'$ are defined by

$$\bar{D}\xi' = u^l \equiv u' + \eta'\bar{u}_y + \zeta'\bar{u}_z, \tag{3.7.1a}$$

$$\bar{D}\eta' = v', \tag{3.7.1b}$$

$$\bar{D}\zeta' = w', \tag{3.7.1c}$$

with $O(\alpha^2)$ error [note that Eq. (3.7.1b) is the same as Eq. (3.6.8)]; furthermore,

$$\overline{\xi'} = \overline{\eta'} = \overline{\zeta'} = 0. \tag{3.7.2}$$

Using the linearized continuity equation

$$u'_x + v'_y + \rho_0^{-1}(\rho_0 w')_z = 0 \tag{3.7.3}$$

[cf. Eqs. (3.2.1d) and (3.4.2d)] together with Eqs. (3.7.1), it can be shown that $\bar{D}[\xi'_x + \eta'_y + \rho_0^{-1}(\rho_0\zeta')_z] = 0$, and thus

$$\xi'_x + \eta'_y + \rho_0^{-1}(\rho_0\zeta')_z = 0, \tag{3.7.4}$$

given suitable initial conditions.

An example of the calculation of $\boldsymbol{\xi}'$ and thus the approximate orbits of fluid particles, given the disturbance velocity (u', v', w'), will be presented in Section 4.5.3 for the case of planetary waves. A knowledge of particle orbits for small-amplitude wave disturbances is useful, for example, in the interpretation of tracer transport in the presence of such waves: see Chapter 9.

The theory given here provides a useful physical interpretation of the Eliassen–Palm flux (Section 3.5) for waves of small amplitude α. Consider a material surface, initially at pressure p_1 and $z_1 \equiv -H\ln(p_1/p_s)$, that is distorted by the waves. The zonal pressure force exerted by the fluid above the surface on that below is

$$F_1 = g^{-1}\overline{p\frac{\partial\Phi}{\partial x}\bigg|_{\text{surface}}} \quad \text{per unit horizontal area,}$$

since $z^* = g^{-1}\Phi$ is the geometric altitude of the surface. Linearizing about z_1 we have

$$F_1 = g^{-1}\overline{p'\Phi'_x}|_{z_1} + O(\alpha^3), \qquad \text{because} \quad \bar{\Phi}_x \equiv 0.$$

Further, since $z_1 + \zeta' = -H \ln[(p_1 + p')/p_s]$ from Eq. (3.1.1), we have $p' = -p_1\zeta'/H + O(\alpha^2)$. Then $F_1 = -\rho_0(z_1)\overline{\zeta'\Phi'_x}$, since $\rho_0(z_1) = p_1/RT_s = p_1/gH$: see Section 3.1.1. For quasi-geostrophic flow, $\Phi'_x = f_0 v'$, from the linearized versions of Eqs. (3.2.3) and (3.2.4), and $\zeta' = -\theta'/\theta_{0z}$ from Eq. (3.7.1c) and the linearized form of Eq. (3.2.9d) if the flow is adiabatic ($Q' = 0$). Hence, $F_1 = \rho_0 f_0 \overline{v'\theta'}/\theta_{0z}$: this equals the quasi-geostrophic expression for $F^{(z)}$, by Eq. (3.5.6).

This result, that $F^{(z)}$ equals the "form drag" per unit horizontal area across an initially isobaric material surface disturbed by small-amplitude adiabatic waves, also holds for flow described by the primitive equations if the waves are steady and frictionless as well. A similar interpretation holds for $F^{(y)}$ in terms of the force across a distorted material surface initially given by y = constant. It follows that $\nabla \cdot \mathbf{F}$ equals the net zonal pressure force, per unit volume in xyz space, on a small, initially zonal material tube of fluid that is distorted by the waves; $\rho_0^{-1}\nabla \cdot \mathbf{F}$ is the corresponding force per unit mass. A finite-amplitude analog of the result for $F^{(z)}$ holds in isentropic coordinates: see Section 3.9.

Small-amplitude theory can also be used to derive an approximation to the Lagrangian mean of any quantity χ. The general definition of the Lagrangian mean of χ is

$$\bar{\chi}^{\mathrm{L}}(\mathbf{x}, t) \equiv \overline{\chi[\mathbf{x} + \boldsymbol{\xi}'(\mathbf{x}, t), t]}, \tag{3.7.5}$$

where $\mathbf{x} + \boldsymbol{\xi}'(\mathbf{x}, t)$ is the current position of the particle whose mean position is $\mathbf{x}$. Application of the identity $\chi \equiv \bar{\chi} + \chi'$ together with a Taylor expansion of Eq. (3.7.5) gives

$$\bar{\chi}^{\mathrm{L}}(\mathbf{x}, t) = \bar{\chi}(\mathbf{x}, t) + \bar{\chi}^{\mathrm{S}}(\mathbf{x}, t) \tag{3.7.6}$$

where the *Stokes correction* $\bar{\chi}^{\mathrm{S}}$ is defined by

$$\bar{\chi}^{\mathrm{S}} = \overline{\boldsymbol{\xi}' \cdot \nabla\chi'} + \tfrac{1}{2}\overline{\xi'_j \xi'_k}\,\frac{\partial^2 \bar{\chi}}{\partial x_j\, \partial x_k} + O(\alpha^3), \tag{3.7.7}$$

and summation over all values of the indices j, k is implied. If χ is a velocity component, say u, $\bar{u}^{\mathrm{S}}$ is known as the Stokes drift, and represents the difference between the Lagrangian-mean velocity $\bar{u}^{\mathrm{L}}$ and the Eulerian-mean velocity $\bar{u}$. The fact that this quantity can be nonzero was pointed out in 1847 by Stokes, who applied a time average to water waves (see also Fig. 3.1). A calculation of the Stokes drift and Lagrangian-mean flow for planetary waves will be given in Section 4.5.3.

3.8 Isentropic Coordinates

3.8.1 The Primitive Equations in Isentropic Coordinates

In this chapter we have up to now been using the log-pressure variable z as a vertical coordinate in the equations of motion. We conclude, however, with a brief discussion of the primitive equations in isentropic coordinates (also called θ coordinates), where the potential temperature (a function of the entropy per unit mass) is used as the vertical coordinate.

The first, and most obvious, reason for the use of isentropic coordinates is that the isentropic "vertical velocity," $D\theta/Dt$, equals the diabatic heating term Q [see Eq. (3.1.3e)]. Thus in adiabatic flow, when $Q = 0$, this "velocity" vanishes, and there is no motion across the isentropic surfaces $\theta =$ constant; isentropic coordinates are therefore partly Lagrangian in character.

In spherical geometry, the primitive equations in θ coordinates are

$$\tilde{D}u - \left(f + \frac{u \tan\phi}{a}\right)v + (a\cos\phi)^{-1}M_\lambda = X - Qu_\theta, \tag{3.8.1a}$$

$$\tilde{D}v + \left(f + \frac{u \tan\phi}{a}\right)u + a^{-1}M_\phi = Y - Qv_\theta, \tag{3.8.1b}$$

$$\sigma_t + (a\cos\phi)^{-1}\{(\sigma u)_\lambda + (\sigma v\cos\phi)_\phi\} = -(\sigma Q)_\theta \tag{3.8.1c}$$

$$M_\theta = \Pi(p) \equiv c_p(p/p_s)^\kappa = c_p e^{-\kappa z/H} \tag{3.8.1d}$$

$$\sigma \equiv -g^{-1}p_\theta. \tag{3.8.1e}$$

Here

$$\tilde{D} \equiv \frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi} \tag{3.8.2}$$

is the time derivative following the components of the flow on an isentrope, and has been used in preference to $D/Dt \equiv \tilde{D} + Q\,\partial/\partial\theta$, so that the nonconservative cross-isentropic advection terms involving $Q\,\partial/\partial\theta$ can be written with the other nonconservative terms on the right of Eqs. (3.8.1a,b). The quantity σ is the "density" in (λ, ϕ, θ) space, as can be shown by considering the mass contained within a volume element lying between isentropes at potential temperatures θ and $\theta + \delta\theta$: see Fig. 3.3. The quantity M is called the Montgomery stream function, and is defined by

$$M \equiv c_p T + \Phi \equiv \theta\Pi(p) + \Phi \tag{3.8.3a,b}$$

where Eq. (3.8.3b) uses Eqs. (1.1.9a) and (3.8.1d); $\Pi(p)$ is known as the Exner function. Subscripts θ denote partial derivatives, and other derivatives here are of course taken at constant θ, not at constant z. The remaining variables are as defined in Section 3.1.1; a brief discussion of the derivation of Eqs. (3.8.1) is given in Appendix 3A. Note incidentally that difficulties arise in regions where $\partial\theta/\partial z^*$ is zero or negative corresponding to neutral or unstable stratification: in such regions θ does not increase monotonically with the geometric height z^*, and σ becomes infinite or negative.

3.8.2 *Ertel's Potential Vorticity in Isentropic Coordinates*

As mentioned at the end of Section 3.7.1, and elsewhere in this book (e.g., Sections 5.2.3, 6.2.4, and 9.1), there are advantages in using θ and Ertel's potential vorticity P as tracers of atmospheric motion. The most convenient way of doing this is by plotting contours of P on isentropic surfaces. In θ coordinates, Ertel's potential vorticity is given by

$$P = \tilde{\zeta}/\sigma \tag{3.8.4a}$$

where

$$\tilde{\zeta} = f - \frac{(u \cos\phi)_\phi}{a \cos\phi} + \frac{v_\lambda}{a \cos\phi}, \tag{3.8.4b}$$

is the vertical component of absolute isentropic vorticity. Using Eqs. (3.8.1) it can be shown that P satisfies

$$\begin{aligned}\tilde{D}P = (\sigma a \cos\phi)^{-1}[&-(X\cos\phi)_\phi + Y_\lambda - Q_\lambda v_\theta \\ &+ Q_\phi u_\theta \cos\phi] + PQ_\theta - QP_\theta.\end{aligned} \tag{3.8.5}$$

Details of the derivation of this equation are given in Appendix 3A, together with a method of verifying that Eq. (3.8.5) reduces to Eq. (3.1.5) on transformation to z coordinates.

Isentropic coordinates provide an enlightening physical explanation of potential vorticity conservation. Consider for simplicity a frictionless ($X = Y = 0$), adiabatic ($Q = 0$) flow, and focus attention on a small material circuit C lying initially on an isentrope (Fig. 3.3). This circuit moves with the fluid but always remains on the same isentrope, since the flow is adiabatic. Its horizontally projected area δA changes according to

$$\frac{D}{Dt}\delta A = (\delta A)\Delta, \tag{3.8.6a}$$

where $\Delta = (a\cos\phi)^{-1}[u_\lambda + (v\cos\phi)_\phi]$ is the isentropic divergence [equal to $u_x + v_y$ in (x, y, θ) coordinates]. Equation (3.8.6a) can be proved most easily for a rectangular circuit $\delta A = \delta x\,\delta y$, using $D\,\delta x/Dt = \delta u \approx u_x\,\delta x$, etc. From Eq. (3.8.1c) with $Q = 0$ (so that $\tilde{D} = D/Dt$) we also have

$$\frac{D\sigma}{Dt} = -\sigma\Delta, \tag{3.8.6b}$$

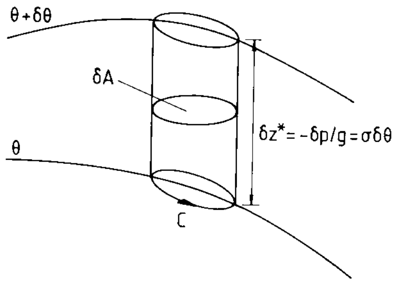

Fig. 3.3. An elementary volume in (x, y, θ) space or (λ, ϕ, θ) space. This has horizontally projected cross-sectional area δA and geometric height δz^*; its mass is therefore $\delta m \equiv \rho\, \delta A\, \delta z^*$, where ρ is the physical density. By the hydrostatic relation of Eq. (1.1.2), $\delta z^* = -g^{-1}\, \delta p$, where δp is the pressure difference between the top and bottom of the element. But

$$-g^{-1}\, \delta p = -g^{-1}\left(\frac{\partial p}{\partial \theta}\right) \delta\theta = \sigma\, \delta\theta \qquad \text{by} \quad \text{Eq. (3.8.1e)}$$

and so $\delta m = \sigma\, \delta A\, \delta\theta$. The quantity σ is therefore the "density" in (λ, ϕ, θ) space. The circuit C is discussed in Section 3.8.2.

while the vorticity equation, obtained by taking the horizontal curl of Eqs. (3.8.1a,b), is

$$\frac{D\tilde{\zeta}}{Dt} = -\tilde{\zeta}\Delta \tag{3.8.6c}$$

[see Eq. (3A.5)]. On eliminating Δ between Eqs. (3.8.6a,b) we obtain

$$\frac{D}{Dt}(\sigma\, \delta A) = 0, \tag{3.8.7a}$$

which expresses conservation of mass per unit θ for the volume depicted in Fig. 3.3. Likewise, Eqs. (3.8.6a,c) give

$$\frac{D}{Dt}(\tilde{\zeta}\, \delta A) = 0: \tag{3.8.7b}$$

this is the Kelvin–Bjerknes circulation theorem for the circuit C and can be thought of as a fluid-dynamical generalization of angular momentum conservation for a rigid body. Since $\sigma\, \delta A$ and $\tilde{\zeta}\, \delta A$ are both conserved following the motion when $X = Y = Q = 0$, their ratio $P = \tilde{\zeta}/\sigma$ is similarly conserved [this also follows from Eqs. (3.8.6b,c)]. Thus conservation of Ertel's potential vorticity is a consequence of the circulation theorem and mass conservation.

A useful alternative version of Eq. (3.8.5) is

$$(\sigma P)_t + (a \cos\phi)^{-1}(\sigma P u - Y + Q v_\theta)_\lambda + (a \cos\phi)^{-1}[(\sigma P v + X - Q u_\theta)\cos\phi]_\phi = 0, \tag{3.8.8}$$

which follows from Eqs. (3A.5) and (3.8.2). This can be regarded either as a vorticity equation [in view of Eq. (3.8.4a)] or as an equation for the mass-weighted potential vorticity σP. The second and third terms in Eq. (3.8.8) form the divergence of a vector flux **L**, which has components

$$\mathbf{L} = (\sigma P u - Y + Q v_\theta, \sigma P v + X - Q u_\theta, 0) \tag{3.8.9}$$

in (λ, ϕ, θ) space. Integration of Eq. (3.8.8) over the whole atmosphere and use of the divergence theorem then shows that sources and sinks for the mass integrated potential vorticity $\iiint \sigma P a^2 \cos\phi \, d\lambda \, d\phi \, d\theta$ can only occur at the ground, and not within the atmosphere itself; frictional and diabatic terms can only help redistribute potential vorticity. [This result can also be demonstrated using the z-coordinate equation, Eq. (3.1.5).] Even more remarkably, the fact that **L** has an identically zero θ component implies that there can be no net transport of potential vorticity across any isentrope, even in the presence of friction and diabatic heating.

3.8.3 Relationships between θ Coordinates and z Coordinates

The primitive equations in θ coordinates bear some formal resemblances to the quasi-geostrophic equations in z coordinates: note for instance the partial analogy between the quasi-geostrophic potential vorticity equation, Eq. (3.2.14), and the Ertel potential vorticity equation, Eq. (3.8.5). However, in several respects the analogy is not complete; for example, D_g involves advection by the geostrophic wind, while $\tilde{D}$ involves advection by the divergent flow (u, v). Another important difference is that the θ-coordinate primitive equations do not require that the potential temperature be always close to the reference profile $\theta_0(z)$ (cf. Section 3.2.3).

A more precise relationship, involving derivatives of q_g and P, was derived by Charney and Stern (1962), who showed that when quasi-geostrophic scaling holds,

$$\left(\frac{\partial P}{\partial s}\right)_{\theta=\text{const.}} \approx \frac{\theta_{0z}}{\rho_0}\left(\frac{\partial q_g}{\partial s}\right)_{z=\text{const.}} \tag{3.8.10}$$

where $s = t, \lambda$, or ϕ (or x or y on a beta-plane); a proof is given in Appendix 3A. In studies of planetary waves (Sections 4.5 and 5.3) and barotropic or baroclinic instability (Section 5.5), the isobaric gradient of q_g plays a central role, and one important use of Eq. (3.8.10) is to suggest interpretations of these phenomena in terms of the distribution of Ertel's potential vorticity on isentropic surfaces.

A major drawback to the use of θ coordinates for prognostic purposes in the troposphere is that the lower boundary condition is generally complicated: the "ground" is not usually an isentrope, and its position in (λ, ϕ, θ)

space is one of the unknowns. (The same difficulty arises in p or z coordinates, but is somewhat less acute unless large-amplitude topography is present.) For modeling the middle atmosphere it may often be sufficient to choose as a lower boundary an isentrope that never intersects the ground (say, the $\theta = 350$ K surface, near the tropopause) and specify suitable conditions there from observations or in idealized form. This parallels the use of a constant-pressure surface as a lower boundary in z coordinates, as mentioned in Section 3.1.2.

3.9 The Zonal-Mean Equations in Isentropic Coordinates

The Eulerian zonal-mean equations in isentropic coordinates have a number of useful features: among other things they bear a close formal resemblance to the *transformed* Eulerian-mean equations in log-pressure coordinates and share the advantages of that set (see Sections 3.5 and 3.6). Moreover, some of the similarities between the primitive equations in θ coordinates and the quasi-geostrophic equations in z coordinates, mentioned in the previous section, carry over to the zonal-mean case, and these can be useful for extending certain quasi-geostrophic results to the primitive equations: an example is given in Section 7.5.

We consider first the zonal momentum equation. This can be derived in "flux form" from Eqs. (3.8.1a,c):

$$\begin{aligned}(\sigma u)_t + (a\cos\phi)^{-1}[(\sigma u^2)_\lambda + (\sigma uv\cos\phi)_\phi] - \sigma(f + ua^{-1}\tan\phi)v \\ + (a\cos\phi)^{-1}\sigma M_\lambda = \sigma X - (\sigma Q u)_\theta. \qquad (3.9.1)\end{aligned}$$

Using Eqs. (3.8.1e) and (3.8.1d), the term σM_λ can be rewritten as follows:

$$\begin{aligned}\sigma M_\lambda &= -g^{-1}p_\theta M_\lambda = -g^{-1}(pM_\lambda)_\theta + g^{-1}pM_{\theta\lambda} \\ &= -g^{-1}(pM_\lambda)_\theta + g^{-1}pp_\lambda\frac{d\Pi}{dp} \\ &= -g^{-1}(pM_\lambda)_\theta + \left[g^{-1}\int_{p_s}^{p} p_1\frac{d\Pi(p_1)}{dp_1}\,dp_1\right]_\lambda. \qquad (3.9.2)\end{aligned}$$

We define an average around a latitude circle on an isentropic surface:

$$\bar{A}(\phi,\theta,t) \equiv (2\pi)^{-1}\int_0^{2\pi} A(\lambda,\phi,\theta,t)\,d\lambda$$

for any field A [contrast the z-coordinate version of Eq. (3.3.1a)] and a deviation $A' \equiv A - \bar{A}$. Substituting from Eq. (3.9.2) into Eq. (3.9.1) and averaging, we obtain

$$\begin{aligned}(\overline{\sigma u})_t + (a\cos^2\phi)^{-1}(\overline{\sigma vu}\cos^2\phi)_\phi - f\overline{\sigma v}\\ -(ga\cos\phi)^{-1}(\overline{p'M'_\lambda})_\theta = \overline{\sigma X} - (\overline{\sigma Q u})_\theta,\end{aligned} \tag{3.9.3}$$

where the two terms in Eq. (3.9.1) involving σuv have been combined and the result $\overline{pM_\lambda} = \overline{pM'_\lambda} = \overline{p'M'_\lambda}$ has been used. Likewise, the zonal average of Eq. (3.8.1c) is

$$\bar\sigma_t + (a\cos\phi)^{-1}(\overline{\sigma v}\cos\phi)_\phi = -(\overline{\sigma Q})_\theta. \tag{3.9.4}$$

We now introduce a mass-weighted zonal mean for any field A,

$$\bar A^* \equiv (\overline{\sigma A})/\bar\sigma, \tag{3.9.5}$$

and put

$$\sigma v = \overline{\sigma v} + (\sigma v)' = \bar\sigma\bar v^* + (\sigma v)',$$

$$\sigma Q = \overline{\sigma Q} + (\sigma Q)' = \bar\sigma\bar Q^* + (\sigma Q)',$$

so that

$$\overline{\sigma vu} = \bar\sigma\bar v^*\bar u + \overline{(\sigma v)'u'}, \tag{3.9.6a}$$

$$\overline{\sigma Qu} = \bar\sigma\bar Q^*\bar u + \overline{(\sigma Q)'u'}. \tag{3.9.6b}$$

On substituting $\overline{\sigma u} = \bar\sigma\bar u + \overline{\sigma' u'}$ in Eq. (3.9.3), subtracting $\bar u \times$ Eq. (3.9.4), using Eqs. (3.9.5) and (3.9.6), and dividing by $\bar\sigma$, we obtain

$$\begin{aligned}\bar u_t + \bar v^*[(a\cos\phi)^{-1}(\bar u\cos\phi)_\phi - f] + \bar Q^*\bar u_\theta - \bar X^*\\ = -\bar\sigma^{-1}(\overline{\sigma' u'})_t + (\bar\sigma a\cos\phi)^{-1}\tilde{\nabla}\cdot\tilde{\mathbf{F}}\end{aligned} \tag{3.9.7a}$$

where $\tilde{\mathbf{F}} = (0, \tilde F^{(\phi)}, \tilde F^{(\theta)})$ is the Eliassen–Palm flux in isentropic coordinates; its components are

$$\tilde F^{(\phi)} = -a\cos\phi\,\overline{(\sigma v)'u'}, \tag{3.9.8a}$$

$$\tilde F^{(\theta)} = g^{-1}\overline{p'M'_\lambda} - a\cos\phi\,\overline{(\sigma Q)'u'}, \tag{3.9.8b}$$

and its isentropic divergence is

$$\tilde\nabla\cdot\tilde{\mathbf{F}} = (a\cos\phi)^{-1}\frac{\partial}{\partial\phi}(\tilde F^{(\phi)}\cos\phi) + \frac{\partial\tilde F(\theta)}{\partial\theta}.$$

The analogy with the z-coordinate TEM equation [Eq. (3.5.2a)] and the definitions in Eq. (3.5.3) should be noted. For small-amplitude disturbances a generalized Eliassen–Palm theorem can be derived, relating $\tilde\nabla\cdot\tilde{\mathbf{F}}$ to wave transience, nonconservative effects, and nonlinearity (cf. Section 3.6). Under nonacceleration conditions it can be shown that $\tilde\nabla\cdot\tilde{\mathbf{F}} \equiv 0$ at finite amplitude as well. Incidentally, the contribution $g^{-1}\overline{p'M'_\lambda}$ to $\tilde F^{(\theta)}$ equals the zonal component of the "pressure torque" (or $a\cos\phi$ times the "form drag" force) per unit horizontal area, exerted by the fluid above an isentrope on that below: cf. Section 3.7.2.

It is straightforward to derive the remaining mean-flow equations in the form

$$\bar{u}(f + \bar{u}a^{-1}\tan\phi) + a^{-1}\bar{M}_\phi = \tilde{G}, \tag{3.9.7b}$$

$$\bar{\sigma}_t + (a\cos\phi)^{-1}(\bar{\sigma}\bar{v}^*\cos\phi)_\phi + (\bar{\sigma}\bar{Q}^*)_\theta = 0, \tag{3.9.7c}$$

$$\bar{M}_\theta - \Pi(\bar{p}) = S \equiv \overline{\Pi(p)} - \Pi(\bar{p}), \tag{3.9.7d}$$

$$\bar{\sigma} = -g^{-1}\bar{p}_\theta. \tag{3.9.7e}$$

Here $\tilde{G}$ represents those terms that lead to departures from gradient-wind balance in Eq. (3.9.7b) [cf. the term G in Eq. (3.5.2b)] and S is a mean quantity of second order in wave amplitude. Analogies with the TEM set [Eq. (3.5.2)] should again be noted.

An alternative form of the mean zonal momentum equation [Eq. (3.9.7a)] that is useful for some purposes (see, e.g., Section 7.5) is

$$\bar{u}_t + \bar{v}^*\{(a\cos\phi)^{-1}(\bar{u}\cos\phi)_\phi - f\} = \bar{\sigma}\overline{\hat{v}\hat{P}}^* + \bar{X} - \overline{Qu_\theta}, \tag{3.9.9}$$

where

$$\hat{A} \equiv A - \bar{A}^* \tag{3.9.10}$$

and P is Ertel's potential vorticity [Eq. (3.8.4)]; the proof is given in Appendix 3A. This equation includes on its right-hand side a force per unit mass proportional to a mean northward advective eddy flux of Ertel's potential vorticity $\overline{\hat{v}\hat{P}}^*$; it is thus analogous to the quasi-geostrophic z-coordinate equation [Eq. (3.5.5a)] with $\rho_0^{-1}\nabla\cdot\mathbf{F}$ replaced by $\overline{v'q'}$, using Eq. (3.5.10). Comparison of Eqs. (3.9.7a) and (3.9.9) and use of Eqs. (3.9.5) and (3.9.10) gives the following relationship between $\overline{\hat{v}\hat{P}}^*$ and the isentropic EP flux divergence:

$$\bar{\sigma}^2\overline{\hat{v}\hat{P}}^* = (a\cos\phi)^{-1}\tilde{\nabla}\cdot\tilde{\mathbf{F}} - (\overline{\sigma'u'})_t + \overline{\sigma'X'} + \bar{\sigma}^2\overline{\hat{Q}(\widehat{u\theta/\sigma})}^* \tag{3.9.11}$$

(Tung, 1986); this is more complicated than the quasi-geostrophic relation in Eq. (3.5.10).

Appendix 3A Derivation of Some Equations in Isentropic Coordinates

3A.1 The Primitive Equations, Eq. (3.8.1)

First, note that, from Eq. (3.8.3b),

$$\frac{\partial M}{\partial\theta} = \Pi(p) + \theta\frac{d\Pi}{dp}\frac{\partial p}{\partial\theta} + \frac{\partial\Phi}{\partial\theta}. \tag{3A.1}$$

But $\theta\, d\Pi/dp = \theta\kappa c_p p^{\kappa-1} p_s^{-\kappa} = RTp^{-1}$ using the definition of $\Pi(p)$ in Eq. (3.8.1d), the fact that $\kappa c_p = R$, and Eq. (1.1.9a). Moreover, $\partial\Phi/\partial\theta = (\partial\Phi/\partial z)(\partial z/\partial\theta) = (RT/H)(-Hp_\theta/p)$ using Eqs. (3.1.3c′) and (3.1.1); Eq. (3.8.1d) follows, since the last two terms in Eq. (3A.1) then cancel. The formula for transforming derivatives gives

$$\left(\frac{\partial M}{\partial s}\right)_{\theta=\text{const.}} = \left(\frac{\partial M}{\partial s}\right)_{z=\text{const.}} - \left(\frac{\partial M}{\partial \theta}\right)\left(\frac{\partial \theta}{\partial s}\right)_{z=\text{const.}}$$

where $s = \lambda$ or ϕ. Using Eqs. (3.8.3a), (3.8.1d), and (1.1.9b), we obtain

$$\left(\frac{\partial M}{\partial s}\right)_{\theta=\text{const.}} = \left(\frac{\partial \Phi}{\partial s}\right)_{z=\text{const.}} + c_p\left(\frac{\partial T}{\partial s}\right)_{z=\text{const.}} - \Pi e^{\kappa z/H}\left(\frac{\partial T}{\partial s}\right)_{z=\text{const.}};$$

by the definition of Π in Eq. (3.8.1d) the last two terms cancel, giving

$$\left(\frac{\partial M}{\partial s}\right)_{\theta=\text{const.}} = \left(\frac{\partial \Phi}{\partial s}\right)_{z=\text{const.}}.$$

Thus Eqs. (3.8.1a,b) follow from Eqs. (3.1.3a,b) on using the fact that $D/Dt \equiv \tilde{D} + Q\,\partial/\partial\theta$ is a coordinate-independent operator. The proof of Eq. (3.8.1c) follows from the conservation of mass, together with the fact that $\delta m = \sigma a^2 \cos\phi\,\delta\lambda\,\delta\phi\,\delta\theta$ is a mass element in (λ, ϕ, θ) space (see the caption of Fig. 3.3). [The analogy between Eq. (3.8.1c) and the compressible-fluid form $\rho_t + \nabla^* \cdot (\rho\mathbf{u}^*) = 0$ in geometric coordinates is immediately evident.]

3A.2 The Potential Vorticity Equation, Eq. (3.8.5)

First define

$$M_1 \equiv M + \tfrac{1}{2}(u^2 + v^2), \tag{3A.2}$$

$$X_1 \equiv X - Qu_\theta, \tag{3A.3a}$$

$$Y_1 \equiv Y - Qv_\theta. \tag{3A.3b}$$

It is then easy to verify that Eqs. (3.8.1a,b) can be written

$$u_t - v\tilde{\zeta} + (a\cos\phi)^{-1}M_{1\lambda} = X_1, \tag{3A.4a}$$

$$v_t + u\tilde{\zeta} + a^{-1}M_{1\phi} = Y_1, \tag{3A.4b}$$

where $\tilde{\zeta}$ is defined in Eq. (3.8.4b). Eliminating M_1 by cross-differentiating Eqs. (3A.4a,b), and rearranging, we can obtain the vorticity equation in the form

$$\tilde{D}\tilde{\zeta} + (a \cos \phi)^{-1}\tilde{\zeta}[u_\lambda + (v \cos \phi)_\phi] = (a \cos \phi)^{-1}[-(X_1 \cos \phi)_\phi + Y_{1\lambda}]; \tag{3A.5}$$

moreover, Eq. (3.8.1c) can be written

$$\tilde{D}\sigma^{-1} - (a \cos \phi)^{-1}\sigma^{-1}[u_\lambda + (v \cos \phi)_\phi] = \sigma^{-2}(\sigma Q)_\theta. \tag{3A.6}$$

But from Eq. (3.8.4a),

$$P = \tilde{\zeta}\sigma^{-1}; \tag{3A.7}$$

hence, using Eqs. (3A.5)-(3A.7) we obtain

$$\tilde{D}P = (\sigma a \cos \phi)^{-1}[-(X_1 \cos \phi)_\phi + Y_{1\lambda}] + P\sigma^{-1}(\sigma Q)_\theta. \tag{3A.8}$$

Using Eqs. (3.8.4) and (3A.3), the right-hand side of Eq. (3A.8) can be rearranged to give Eq. (3.8.5).

To transform Eq. (3.8.5) to obtain the z-coordinate version [Eq. (3.1.5)] it is helpful to observe that

$$\frac{\sigma}{\rho_0} = \frac{\partial z}{\partial \theta} = \frac{\partial(\lambda, \phi, z)}{\partial(\lambda, \phi, \theta)} \tag{3A.9}$$

and that, for example, $\partial Y/\partial\lambda$ at constant θ can be written $\partial(Y, \phi, \theta)/\partial(\lambda, \phi, \theta)$. The rule for multiplication of Jacobians then gives

$$\sigma^{-1}\left(\frac{\partial Y}{\partial \lambda}\right)_{\theta=\text{const.}} = \rho_0^{-1}\frac{\partial(Y, \phi, \theta)}{\partial(\lambda, \phi, z)} = \rho_0^{-1}\frac{\partial(Y, \theta)}{\partial(\lambda, z)}.$$

The other Jacobian terms in Eq. (3.1.5) are obtained in a similar manner.

3A.3 Charney and Stern's Relation, Eq. (3.8.10)

From Eq. (3A.7) and the left-hand equality in Eq. (3A.9) we have

$$P = \rho_0^{-1}\theta_z\tilde{\zeta}; \tag{3A.10}$$

then from the usual formula for transforming derivatives,

$$\left(\frac{\partial P}{\partial s}\right)_{\theta=\text{const.}} = P_s - \theta_z^{-1}P_z\theta_s$$

(where the subscript s here denotes a derivative at constant z), it can easily be verified that

$$\left(\frac{\partial P}{\partial s}\right)_{\theta=\text{const.}} = \theta_z(P\theta_z^{-1})_s + P^2\theta_z^{-1}(P^{-1}\theta_s)_z.$$

Using Eq. (3A.10) this yields

$$\left(\frac{\partial P}{\partial s}\right)_{\theta=\text{const.}} = \frac{\theta_z}{\rho_0}\left[\tilde{\zeta}_s + \frac{\tilde{\zeta}^2}{\rho_0}\left(\frac{\rho_0\theta_s}{\tilde{\zeta}\theta_z}\right)_z\right], \tag{3A.11}$$

since $\rho_0 = \rho_0(z)$. Equation (3A.11) is exact, but on using the leading approximations $\theta_z \approx \theta_{0z}$, $\tilde{\zeta}_s \approx \zeta_{gs}$, $\tilde{\zeta} \approx f_0$ we obtain the quasi-geostrophic result

$$\left(\frac{\partial P}{\partial s}\right)_{\theta=\text{const.}} \approx \frac{\theta_{0z}}{\rho_0}\left[\zeta_{gs} + \frac{f_0}{\rho_0}\left(\frac{\rho_0\theta_s}{\theta_{0z}}\right)_z\right],$$

whence Eq. (3.8.10) follows on using Eqs. (3.2.5), (3.2.15a), and the fact that ρ_0 and θ_0 are independent of s.

3A.4 The Mean Momentum Equation in the Form of Eq. (3.9.9)

The zonal mean of Eq. (3A.4a) is

$$\bar{u}_t - \overline{v\tilde{\zeta}} = \bar{X}_1 \tag{3A.12}$$

and

$$\overline{v\tilde{\zeta}} = \overline{\sigma v P} = \bar{\sigma}(\overline{vP})^* \tag{3A.13}$$

by Eqs. (3A.7) and (3.9.5). Now using Eqs. (3.9.5) and (3.9.10) it can be verified that $\overline{AB}^* = \bar{A}^*\bar{B}^* + \overline{\hat{A}\hat{B}}^*$ for any fields A, B. Thus

$$\bar{\sigma}(\overline{vP})^* = \bar{\sigma}\bar{v}^*\bar{P}^* + \bar{\sigma}\overline{\hat{v}\hat{P}}^* = \bar{v}^*\bar{\tilde{\zeta}} + \bar{\sigma}\overline{\hat{v}\hat{P}}^* \tag{3A.14}$$

by Eqs. (3.9.5) and (3A.7). Then Eq. (3.9.9) follows from Eqs. (3A.12)–(3A.14), (3.8.4b), and (3A.3a). Note incidentally that Eq. (3A.12) can be written as $\bar{u}_t = \overline{L^{(\phi)}}$, by Eqs. (3A.3a) and (3A.13), where $L^{(\phi)}$ is the ϕ component of the potential vorticity flux vector defined in Eq. (3.8.9).

Appendix 3B Boundary Conditions on the Residual Circulation

We present here examples of the kind of boundary conditions that may apply to $\bar{v}^*$ and $\bar{w}^*$.

3B.1 Side Boundaries

In a channel with vertical sidewalls, $v = 0$ on these walls, as in Section 3.1.2.c. Thus $\bar{v} = v' = 0$ on these walls, and hence $\bar{v}^* = 0$ there, by Eq. (3.5.1a).

3B.2 Lower Boundary

For simplicity, we concentrate on the case in which the log-pressure on a lower material boundary is specified, and use Cartesian coordinates. Thus,

$$w = \frac{D\zeta}{Dt} \quad \text{at} \quad z = \zeta(x, y, t) \tag{3B.1}$$

by Eq. (3.1.6c), when ζ is given. We suppose that $\zeta'(x, y, t)$ is $O(\alpha)$, where $\alpha \ll 1$, and that $\bar{\zeta}(y, t)$ is $O(\alpha^2)$. Consistent with these, we take $\bar{u} = O(1)$, $\bar{v}, \bar{w} = O(\alpha^2)$ and $(u', v', w') = O(\alpha)$. Then at $O(\alpha)$, Eq. (3B.1) gives

$$w' = \bar{D}\zeta' \quad \text{at} \quad z = \bar{\zeta}, \tag{3B.2}$$

where $\bar{D} = \partial/\partial t + \bar{u}\,\partial/\partial x$. Note that this applies at the "mean" value $\bar{\zeta}$ of the log-pressure altitude of the boundary.

At $O(\alpha^2)$, Eq. (3B.1) gives

$$\bar{w} + \overline{\zeta' w'_z} = \bar{\zeta}_t + \overline{u'\zeta'_x} + \overline{v'\zeta'_y} \quad \text{at} \quad z = \bar{\zeta} \tag{3B.3}$$

after zonal averaging; the second term on the left comes from Taylor-expanding $w(x, y, \zeta, t)$ about $z = \bar{\zeta}$. Now the Cartesian form of the continuity equation, Eq. (3.4.2d), can be written

$$u'_x + v'_y + w'_z = w'/H, \tag{3B.4}$$

since $\rho_{0z}/\rho_0 = -H^{-1}$. Rearranging Eq. (3B.3) and using Eqs. (3B.4), (3B.2), and the fact that $\overline{(\cdots)_x} \equiv 0$, we obtain

$$\bar{w} = \bar{\zeta}_t + (\overline{v'\zeta'})_y - (2H)^{-1}(\overline{\zeta'^2})_t \quad \text{at} \quad z = \bar{\zeta}; \tag{3B.5}$$

using the Cartesian form of Eq. (3.5.1b), this gives

$$\bar{w}^* = \bar{\zeta}_t + \overline{[v'(\zeta' + \theta'/\bar{\theta}_z)]}_y - (2H)^{-1}(\overline{\zeta'^2})_t \quad \text{at} \quad z = \bar{\zeta}. \tag{3B.6}$$

From the definition in Eq. (3.6.8) [or Eq. (3.7.1b)] of the northward parcel displacement η', we have,

$$\begin{aligned}\overline{v'(\zeta' + \theta'/\bar{\theta}_z)} &= \overline{[\eta'(\zeta' + \theta'/\bar{\theta}_z)]}_t - \overline{\eta'\bar{D}(\zeta' + \theta'/\bar{\theta}_z)},\\ &= \overline{[\eta'(\zeta' + \theta'/\bar{\theta}_z)]}_t - \overline{\eta'(Q' - v'\bar{\theta}_y)}/\bar{\theta}_z \end{aligned} \tag{3B.7}$$

at $z = \bar{\zeta}$, by the Cartesian form of Eq. (3.4.2e), and Eq. (3B.2). Thus, from Eqs. (3B.6), (3B.7), and (3.6.8),

$$\begin{aligned} \bar{w}^* = \bar{\zeta}_t &+ [\overline{\eta'(\zeta' + \theta'/\bar{\theta}_z)}]_{yt} + \tfrac{1}{2}[\overline{\eta'^2}\,\bar{\theta}_y/\bar{\theta}_z]_{yt} - (\overline{\eta' Q'}/\bar{\theta}_z)_y \\ &- (2H)^{-1}(\overline{\zeta'^2})_t + O(\alpha^3) \qquad \text{at} \quad z = \bar{\zeta}. \end{aligned} \tag{3B.8}$$

In particular, if $\bar{\zeta}_t = 0$, then $\bar{w}^* = O(\alpha^3)$ at $z = \bar{\zeta}$ for steady, adiabatic disturbances; however, $\bar{w} = (\overline{v'\zeta'})_y$ in this case, and is generally $O(\alpha^2)$. Thus $\bar{w}^*$ is negligible but $\bar{w}$ is generally not negligible at $z = \bar{\zeta}$, to second order in wave amplitude.

References

3.1. Comprehensive accounts of the principles of fluid dynamics are given in many texts, such as that of Batchelor (1967). Careful discussions of the dynamical and geometric arguments involved in the derivation of the primitive equations are given by Phillips (1973) and Gill (1982), Section 4.12. Hoskins *et al.* (1985) present a historical account of the use of Ertel's potential vorticity in meteorology, and mention several modern applications. The derivation of the kinematic boundary condition at a material surface is explained by Batchelor (1967), p. 73.

3.2. Formal derivations of the quasi-geostrophic equations on a beta-plane are given for example by Pedlosky (1979) and Gill (1982). The omega equation is discussed by Hoskins, Dragici and Davies (1978).

3.5. The transformation [Eq. (3.5.1)] leading to the TEM equations [Eqs. (3.5.2)] was introduced by Andrews and McIntyre (1976a, 1978a) and Boyd (1976). Alternative definitions of the residual circulation are given by Andrews and McIntyre (1978a) and Holton (1981). The identity of Eq. (3.5.10) is due to Bretherton (1966a).

3.6. The significance of conservation laws such as Eq. (3.6.2) is discussed by McIntyre (1980a, 1981). Successive generalizations of the original Charney-Drazin nonacceleration theorem were derived by Dickinson (1969), Holton (1974, 1975), Boyd (1976), and Andrews and McIntyre (1976a, 1978a).

3.7. Finite-amplitude versions of the generalized Eliassen-Palm theorem are discussed by Edmon *et al.* (1980), Andrews (1983) and Killworth and McIntyre (1985). The GLM theory is described in detail by Andrews and McIntyre (1987b,c); a useful introductory survey is that of McIntyre (1980a). Modified versions of this theory are outlined by McIntyre (1980a,b). For the relationship between the "form drag" and the EP flux see Bretherton (1969) and Andrews and McIntyre (1976a).

3.8. The primitive equations in θ coordinates are derived for example by Dutton (1976). Implications of the form of Eq. (3.8.8) of the potential vorticity conservation law are discussed by Haynes and McIntyre (1987).

3.9. The approach adopted here is a development of that due to Andrews (1983); see also Tung (1986).