Modelling convective scale background error covariances using normal modes

Ruth Petrie\(^1\) and Ross Bannister\(^1\)

\(^1\)Department of Meteorology
University of Reading

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Modelling the $B$-matrix

The $B$ matrix can be modelled via a Control Variable transform (CVT)

$$\delta \mathbf{x} = \mathbf{U} \delta \chi. \quad (1)$$

Substituting this into the incremental background term:

$$J_b(\delta \chi) = \delta \chi^T \mathbf{U}^T \mathbf{B}^{-1} \mathbf{U} \delta \chi, \quad (2)$$

then by choosing $\mathbf{U}$ such that:

$$\mathbf{U}^T \mathbf{B}^{-1} \mathbf{U} = \mathbf{I}, \quad (3)$$

results in a simplified background term:

$$J_b(\delta \chi) = \delta \chi^T \delta \chi. \quad (4)$$

Implied $B$ matrix is

$$B^i = \mathbf{U} \mathbf{U}^T \quad (5)$$
How $B$ is represented in reality

Typical structure of $U$

- Parameter transform decorrelates multivariate relationships.
- Spatial transforms decorrelates univariate relationships.
- Variance scaling ensures $B$ is the identity in control space.

Multivariate aspects

- Typically transform to variables which are assumed to be uncorrelated.
- Assume errors in balanced variables are uncorrelated from errors in unbalanced variables.
- A mass-wind relationship is used to described balanced flow.
At the convective scale:

- the Rossby number is not small and the geostrophic relationship may not be valid.

- the flow is non-hydrostatic and acoustic modes are present.
CVTs at the convective scale

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An alternative is to use the normal modes of the linearized equations.

- NMs are independent by definition.

- NMs have been used to describe forecast error covariances in the tropics. (Žagar et al, 2004)
Model equations
Model equations

Starting from the standard Euler equations:

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p + g \mathbf{k} + \mathbf{f} \times \mathbf{u} = 0, \quad (6a) \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (6b) \]

\[ \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = 0, \quad (6c) \]

\[ p = \rho R \left( \frac{p}{p_{00}} \right)^{\kappa} \theta. \quad (6d) \]

Assume:
- laterally periodic,
- no slip, and rigid lid for the vertical boundary conditions,
- homogeneity in the \( y \)-direction,
- \( f \)-plane,
- \( \phi = \phi_0(z) + \phi'(x, z) \) and \( \theta = \theta_R + \theta_0(z) + \theta'(x, z) \).
Approximations

- Impose basic state satisfies hydrostatic balance and the equation of state:

\[
\frac{\partial p_0}{\partial z} = -\rho_0 g, \quad p_0 = \rho_0 R \left( \frac{p_0}{p_{00}} \right)^\kappa (\theta_R + \theta_0).
\]  

(7)

- Define the Brunt Väisälä frequency:

\[
N^2 = \frac{g}{\theta_R} \frac{d\theta_0}{dz}.
\]

(8)

- Define buoyancy:

\[
b = b_0(z) + b' = \frac{g}{\theta_R} (\theta_0(z) + \theta').
\]

(9)

- Make the Boussinesq approximation (i.e. neglect density perturbations except when multiplied by \(g\)).
Simplifications
Simplifications

- Let $N^2$ be a tuneable parameter $A^2$.
- Multiply all advective terms by $B$ (another tuneable parameter).
- Approximate:
  \[
  \frac{\theta'}{\theta_R} = -\frac{\rho'}{\rho_0}. \tag{10}
  \]
- Adopt simplified equation of state:
  \[p = C\rho \quad \text{where} \quad C \text{ is a constant.} \tag{11}\]
- Scale density: $\rho = \rho_0(z) + \rho'$
  (1) by $\rho_0(z) \rightarrow \tilde{\rho} = 1 + \tilde{\rho}'$
  (2) by $C \rightarrow \tilde{\rho} = C + \tilde{\rho}'$
  drop the $\sim$ notation.
Toy model equations

\[
\frac{\partial u}{\partial t} + B \mathbf{u} \cdot \nabla u + \frac{\partial p'}{\partial x} - fv = 0, \tag{12a}
\]

\[
\frac{\partial v}{\partial t} + B \mathbf{u} \cdot \nabla v + fu = 0, \tag{12b}
\]

\[
\frac{\partial w}{\partial t} + B \mathbf{u} \cdot \nabla w + \frac{\partial p'}{\partial z} - b' = 0, \tag{12c}
\]

\[
\frac{\partial p'}{\partial t} + B \nabla \cdot (\rho \mathbf{u}) = 0, \tag{12d}
\]

\[
\frac{\partial b'}{\partial t} + B \mathbf{u} \cdot \nabla b' + A^2 w = 0. \tag{12e}
\]

- Model conserves energy analytically.
- 360 longitudinal points, \( \delta x = 1.5 \text{km} \).
- 60 vertical levels, \( \delta z \sim 260 \text{m} \).
- Centered-in-time, forward-backward (Cullen and Davies, 1991).
Linear analysis results

Acoustic wave speed sensitivity to $BC; A^2 = 4 \times 10^{-4} \text{ s}^{-2}$

Gravity sensitivity to $A^2; BC = 10^5 \text{ m}^2\text{s}^{-2}$
**Convective-like behaviour**

\( w \) - vertical component of wind

\[
A^2 = 4 \times 10^{-4}, \\
B = 10^{-2}, \quad C = 10^4
\]

\[
A^2 = 4 \times 10^{-5}, \\
B = 10^{-2}, \quad C = 10^4
\]
Geostrophic adjustment

Initial condition

1/2 hour

1 hour

3 hours
Structure of NMCVT

\[ \delta \chi = U_I^{-1} U_E^{-1} U_S^{-1} U_V^{-1} U_M^{-1} U_H^{-1} \delta x \]  \hspace{1cm} (13)

- \( U_H^{-1} \) is a horizontal Fourier transform.
- \( U_M^{-1} \) is a Helmholtz variable transform.
- \( U_V^{-1} \) is a vertical Fourier transform.
- \( U_S^{-1} \) is a symmetric scaling.
- \( U_E^{-1} \) is a projection on to eigenvectors.
- \( U_I^{-1} \) is a scaling to ensure unit variance.
Normal modes

Parameters

\[ A = 4 \times 10^{-4} \]
\[ B = 10^{-1} \]
\[ C = 10^4 \]
Covariance spectrum

physical mode 1: acoustic

physical mode 2: gravity

physical mode 3: balanced

physical mode 5: acoustic

physical mode 4: gravity

m − vertical wave no.

variance

$k = 1$

$k = 40$

$k = 80$
Implied covariances: including all modes

Structure function of $r'(100,30)
positive perturbation
All Modes
ensemble 14; $A=10^{-4}$, $B=0.1$ $C=10^4$
Ensemble derived covariances

Ensemble 14
2 hour forecast covariances with $r^*$ at $x = 180$, $z = 30$
Implied covariances: balanced mode only

\[ p' \]

\[ v \]

\begin{align*}
\text{Height (m)} & \quad \text{Longitude (km)} \\
14000 & \quad 100 \\
12000 & \quad 200 \\
10000 & \quad 300 \\
8000 & \quad 400 \\
6000 & \quad 500 \\
4000 & \\
2000 & \\
100 & \\
0 & \\
-100 & \\
-200 & \\
-300 & \\
-400 & \\
-500 & \\
\end{align*}

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100 & \\
0 & \\
-100 & \\
-200 & \\
-300 & \\
-400 & \\
-500 & \\
\end{align*}
Implied covariances: gravity modes only

\[ \begin{align*}
\text{p'} & \quad \begin{array}{c}
\text{v} \\
\text{b'}
\end{array} & \quad \begin{array}{c}
\text{v} \\
\text{p'}
\end{array}
\end{align*} \]
Implied covariances: acoustic modes only
Future work

- Investigate the degree to which the linearised normal modes are uncorrelated in the non-linear system.
- Compare the NMCVT with a standard approach.
- Assess the impact of the NMCVT inside an assimilation system.
- Hybrid methods.
- What covariances are appropriate at the convective scale?
A non-hydrostatic toy model has been developed as a tool to investigate the convective scale data assimilation problem.

A normal mode approach to covariance modelling has been adopted.

Both the model and the covariance model should be a useful tool in further investigating the convective scale data assimilation problem.
Thank you for your attention, any questions/comments?
Eigenvectors

acoustic modes

gravity modes

balanced mode