An idealised fluid model for inexpensive DA experiments and its relevance for NWP

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High-resolution (convective-scale) NWP models are becoming the norm

- more dynamical processes such as convection, cloud formation, and small-scale gravity waves, are resolved explicitly

DA techniques need to evolve in order to keep up with the developments in high-resolution NWP

- breakdown of dynamical balances (e.g., hydrostatic and semi/quasi-geostrophic) at smaller scales
- strongly nonlinear processes associated with convection and moisture/precipitation
- move towards ensemble-based methods
It may be unfeasible, and indeed undesirable, to investigate the potential of DA schemes on state-of-the-art NWP models. Instead idealised models can be employed that:

- capture some fundamental processes
- are computationally inexpensive to implement
- allow an extensive investigation of a forecast/assimilation system in a controlled environment
Using idealised models

It may be unfeasible, and indeed undesirable, to investigate the potential of DA schemes on state-of-the-art NWP models. Instead idealised models can be employed that:

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‘Toy’ models:

- Lorenz (L63, L95, L2005, ... )
- BV/QG models (Bokhove et al., poster this workshop)
- simplified NWP models
Using idealised models: approach

1. Describe a physically plausible idealised model and implement numerically.
   ▶ based on the shallow water equations (SWEs).
   ▶ compare dynamics of the modified model to those of the classical shallow water theory

2. Ensemble-based DA - relevant for convective-scale NWP?
   ▶ initial perturbations to represent forecast error
   ▶ “tuning” the observing system and the observational influence diagnostic

3. Current/future work and ideas.
   ▶ DA: a comparison with VAR
   ▶ advanced numerics: non-negativity of ‘rain’
   ▶ other fluid dynamical models
   ▶ which characteristics of NWP can we seek to replicate in idealised models?
1. SWEs: an extension

**Aim:** modify the SWEs to include more complex dynamics relevant for the 'convective-scale', extending the model employed by Würsch and Craig (2014).

- convective updrafts - artificially mimic *conditional instability* (positive buoyancy)
- idealised representation of precipitation, including source and sink.
- contain *switches* for the onset of convection and precipitation - realistic (and highly *nonlinear*) features of operational NWP models.
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2D rotating SWEs on an \( f \)-plane with no variation in the \( y \)-direction (\( \partial_y = 0 \)):

\[
\begin{align*}
\partial_t h + \partial_x (hu) &= 0, \\
\partial_t (hu) + \partial_x (hu^2 + p(h)) - fhv &= -gh\partial_x b, \\
\partial_t (hv) + \partial_x (huv) + fhv &= 0, \\
\partial_t b &= 0,
\end{align*}
\]

where \( p(h) \) is an effective pressure: \( p(h) = \frac{1}{2}gh^2 \).
Modified SWEs

Ingredients:

- two threshold heights $H_c < H_r$: when fluid exceeds these heights, different mechanisms kick in and alter the classical SW dynamics.
- modifications to the effective pressure gradient (equivalently, geopotential gradient) in the momentum equation.
- extra equation for the conservation of model ‘rain’ to close the system.
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\[
\partial_t h + \partial_x (hu) = 0,
\partial_t (hu) + \partial_x (hu^2 + p(h)) + hc_0^2 \partial_x r - fhv = -gh \partial_x b,
\partial_t (hv) + \partial_x (huv) + fhu = 0,
\partial_t (hr) + \partial_x (hur) + h\tilde{\beta} \partial_x u + \alpha hr = 0,
\partial_t b = 0,
\]

where \( p(h) = \begin{cases} 
\frac{1}{2} gH_c^2, & \text{for } h + b > H_c, \\
\frac{1}{2} gh^2, & \text{otherwise},
\end{cases} \)

and \( \tilde{\beta} = \begin{cases} 
\beta, & \text{for } h + b > H_r, \partial_x u < 0, \\
0, & \text{otherwise}.
\end{cases} \)
Some theoretical aspects

- Shallow water systems are **hyperbolic**, and can thus be solved via a range of numerical recipes for hyperbolic systems. What about the modified system?
- Vector formulation:

\[ \partial_t U + \partial_x F(U) + G(U)\partial_x U + S(U) = 0 \]

- Hyperbolicity determined by eigenstructure (**all eigenvalues must be real**). Eigenvalues of the system are determined by the matrix:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-u^2 - c_0^2 r + \partial_h p & 2u & c_0^2 & 0 & gh \\
-\beta + r & \beta + r & u & 0 & 0 \\
-uv & v & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
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\frac{\partial F}{\partial U} + G(U) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-u^2 - c_0^2 \beta + \partial_h p & 2u & c_0^2 & 0 & gh \\
-u(\beta + r) & -u \beta + r & u & 0 & 0 \\
-uv & v & 0 & u & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

- This matrix has five eigenvalues:

\[
\lambda_{1,2} = u \pm \sqrt{\partial_h p + c_0^2 \beta}, \quad \lambda_{3,4} = u, \quad \text{and} \quad \lambda_5 = 0,
\]

- Since \(p(h)\) is non-decreasing and \(\tilde{\beta}\) non-negative, the eigenvalues are real. Hence, the modified SW model is hyperbolic.
Numerics

Scheme:

- large literature on numerical routines for hyperbolic systems of PDEs.
- Rhebergen et al. (2008) developed a novel discontinuous Galerkin (DG) finite element framework for hyperbolic system of PDEs with non-conservative products $G(U)\partial_x U$.
- in most simple case (DG0), analagous to Godunov’s FV scheme in which a numerical flux must be evaluated

$$\frac{d}{dt} U_k + \frac{1}{\Delta x_k} \left[ P^{NC}(U_k, U_{k+1}) - P^{NC}(U_{k-1}, U_k) \right] + \frac{S(U_k)}{\Delta x_k} = 0.$$
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$$

Experiments:
- Rossby geostrophic adjustment in a periodic domain
- describes the evolution of the free surface height $h$ when disturbed from its rest state by a transverse jet, i.e., fluid with an initial constant height profile is subject to a localised $v$-velocity distribution.
- non-dimensional parameters: $Ro = 1$ and $Fr = 2$. 
Adjustment of a transverse jet in RSW

Below $H_c$ and $H_r$:

Above $H_c$ but below $H_r$:

Above $H_c$ and $H_r$:

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Modified SW model for DA
2. Ensemble-based DA for idealised models

Ensemble Kalman filter: twin model setting

- Imperfect model:
  - "truth" trajectory: run at high resolution
  - "forecast" model: run at lower resolution at which small-scale features (e.g., localised moisture transport) are not fully resolved
  - ensemble (covariance) inflation \((x_i^f \leftarrow \gamma(x_i^f - \bar{x}^f) + \bar{x}^f)\) applied to account for the model error due to resolution mismatch
  - localisation \((P^f \leftarrow \rho_{loc} \circ P^f)\) applied to damp spurious long-range correlations
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- “tuning” the observing system: what to observe? how often? with how much noise?

- observational influence diagnostic (after Cardinali et al. (2004)) averaged over cycles:

\[ OI = \frac{tr(HK)}{p} \]
Before assimilating...: ensemble spread as a representation of forecast error

RMS spread and RMS error of ensemble mean

Ratio of ensemble spread ($N = 100$) to forecast error:
Cycled assimilation...: how does an analysis look?

Field-averaged RMS errors after an analysis cycle (Obs. error = 0.1):

<table>
<thead>
<tr>
<th></th>
<th>Forecast</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.0731</td>
<td>0.0725</td>
</tr>
<tr>
<td>$hu$</td>
<td>0.1052</td>
<td>0.0812</td>
</tr>
<tr>
<td>$hv$</td>
<td>0.1374</td>
<td>0.0696</td>
</tr>
<tr>
<td>$hr$</td>
<td><strong>0.0169</strong></td>
<td>0.0238</td>
</tr>
</tbody>
</table>

Observational influence diagnostic:

$$OI = \frac{tr(HK)}{p} = 0.28$$
Cycled assimilation...: how does an analysis look?

Field-averaged RMS errors after an analysis cycle (Obs. error = 0.05):

<table>
<thead>
<tr>
<th></th>
<th>Forecast</th>
<th>Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.0828</td>
<td>0.0816</td>
</tr>
<tr>
<td>$h u$</td>
<td>0.0991</td>
<td>0.0906</td>
</tr>
<tr>
<td>$h v$</td>
<td>0.1297</td>
<td>0.0793</td>
</tr>
<tr>
<td>$h r$</td>
<td>0.0200</td>
<td>0.0293</td>
</tr>
</tbody>
</table>

Observational influence diagnostic:

$$OI = \frac{tr(HK)}{p} = 0.42$$
Lots of parameters and different set-ups to explore and play with:

- observe only one variable (e.g., the height field) and compare; or observe nonlinearly (e.g., radial wind)
- include topography and observe downstream of a mountain
- increase the ratio of truth to forecast resolution to observe smaller-scale features
- (too) many more possibilities...
3. Current/future work and ideas

**DA:**
- setting up a demonstration system that compares EnKF with VAR in which B matrix is derived from ensemble.

**Numerics:**
- extension to ensure non-negativity of $hr$, à la Audusse et al., 2004.

\[
P^{NC}(U_k, U_{k+1}) \rightarrow P^{NC}(U_{(k+1)/2}^-, U_{(k+1)/2}^+)
\]

- reconstructed states $U_{(k+1)/2}^\pm$ impose that $h$ and $hr$ cannot become negative yet dry states $hr = 0$ can be computed (given a derived time-step criterion).

**Other models of interest:**
- (dimensionally-reduced) adapted moist Boussinesq shallow water equations (after Zerroukat and Allen, 2015)
- 3D QG model with anisotropic rotating convection (Bokhove et al., poster)
3. Current/future work and ideas

Other diagnostics and the question of ‘relevance’:

- how can findings based on ‘toy’ models generalise to and provide useful insight for operational NWP forecast/assimilation systems?

- observational influence diagnostic:
  - global NWP: 0.15 (Cardinali et al., 2004)
  - convective-scale NWP: 0.2 - 0.5? (Brousseau et al., 2014)
Other diagnostics and the question of ‘relevance’:

- how can findings based on ‘toy’ models generalise to and provide useful insight for operational NWP forecast/assimilation systems?
- observational influence diagnostic:
  - global NWP: 0.15 (Cardinali et al., 2004)
  - convective-scale NWP: 0.2 - 0.5? (Brousseau et al., 2014)
- error-growth properties of the idealised model should be similar to those in operational models:
  - error-growth characteristics of assimilating model determine magnitude and structure of the updated $P^f$ represented by the ensemble.
  - error-doubling time for forecast error for global NWP known to be on the order of days - what about convective-scale?
Summary and outlook

- novel fluid dynamical models to fill the ‘complexity gap’ between ODE models and the primitive equations / state-of-the-art NWP models
- Idealised convective-scale DA experiments with characteristics relevant for NWP
- Implement a variational algorithm (in which initial covariance comes from the ensemble)
- Integrate model(s) into Met Office’s nascent ‘VarPy’ framework as a repository for idealised DA experiments
Thank you very much for your attention.

References:

- Brousseau et al., 2014: A posteriori diagnostics of the impact of observations on the AROME-France convective-scale data assimilation system. *QJRMS*, 140(680), 982-994.