On a 3D Model with Anisotropic, Rotating Convection and Phase Changes for DA

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1 Introduction

• High-resolution (convective-scale) NWP models are becoming the norm: more dynamical processes such as convection, cloud formation, & small-scale gravity waves, are resolved explicitly.

• DA techniques need to evolve in order to keep up with the developments in high-resolution NWP.

• It may be feasible (and even undesirable) to investigate the potential of DA schemes on state-of-the-art NWP models. Idealised models have been employed that:
  - Capture some fundamental features of dynamics, are computationally inexpensive to implement, and allow an extensive investigation of the proposed scheme.
  - A hierarchy of "toy" models, e.g., Lorenz model (1963), OG/BV model have been employed in DA, including a "convective-scale" 1D shallow-layer model (Kent et al., 2015, this DA workshop).

• Here, we propose to add a 3D rotating convection model to this hierarchy.

2 Boussinesq Parent Model

• Following Julien et al. (2006), rotating Boussinesq equations are scaled with $\nu = \nu_b$ buoyancy $B = \frac{b_0}{\rho_0} \rho / \rho_0$, horizontal and vertical length scales $L$ and $z_L \equiv \frac{\rho_0}{\rho_b}$, and velocity scales $U_L$ and $z_L$.

• The resulting dimensionless model reads:

$$
\frac{D u}{D t} + \nabla \cdot (u \otimes u) - \nabla \psi + \frac{1}{\rho_0} \nabla z = 0 
$$

$$
\frac{D b}{D t} = - \nabla \cdot (b \otimes u) + \frac{1}{\rho_0} \nabla \delta z b 
$$

with velocity $u = (-u_z, 0, b)$, buoyancy $b = -\rho/\rho_b$, background density $\rho_b$, and Boussinesq buoyancy $B$.

• Rosnay $Ro = U(2)/BL$, Froude $Fr = U(2)/H_L$ (buoyancy frequency $N_0$, Euler $\nu = \frac{\nu_b}{\nu}, \rho_0 \approx \rho_b$, buoyancy $\beta = 0$), and Reynolds $Re = \frac{U_k^2}{\nu}$, for fixed $D$ and $P$ numbers.

3 3D Model of Rotationally Constrained Convection

• Using a multi-scale, singular expansion in $Re = \epsilon$ with $P = \frac{1}{\nu_b}$ and $T = \epsilon (A, \nu_b, \rho_0 \approx \rho_b)$, $\delta b = \delta b_0 + \delta b_1$, a non-hydrostatic rotationally constrained model Julien et al. (2006) derive is:

$$
\partial_z \zeta = -J(\psi, \zeta) + \partial_z u_0 + \frac{1}{\rho_0} \nabla z \delta z + \frac{1}{\rho_b} \nabla \delta \psi 
$$

$$
\partial_z \psi = -J(\psi, \zeta) + \partial_z \psi_0 + \frac{1}{\rho_0} \nabla z \delta z + \frac{1}{\rho_b} \nabla \delta \psi + \frac{1}{\rho_b} \nabla \delta \psi 
$$

$$
\partial_z \delta \psi = -J(\psi, \zeta) + \partial_z \delta \psi_0 + \frac{1}{\rho_0} \nabla z \delta z + \frac{1}{\rho_b} \nabla \delta \psi + \frac{1}{\rho_b} \nabla \delta \psi 
$$

• with horizontal Laplacian $\nabla^2 \equiv \partial_x^2 + \partial_y^2$, (leading order) vertical velocity $u_0$,

• slowly evolving or constant buoyancy $b_0$, next order buoyancy $b_1$, $b \approx -\psi$, Jacobian $J(\psi, \zeta, \psi_0, \zeta_0)$, and

• underlined terms denote the dissipative, viscous terms (or turbulent counterparts).

• We consider a cylindrical domain $D = \{r < \sqrt{\alpha + \beta}, \theta \in [0, 2\pi]\}$, and, on average, a vertical coordinate $z \in [h_1, h_2]$ for fixed $H$ and $H_T$.

4 3D Baroclinic Quasigeostrophy

• Ignoring the underlined dissipative terms in (2), stratified quasigeostrophy arrives when hydrostatic balance is assumed (equating the twice underlined terms), and the buoyancy equation is used to eliminate $b_0 \equiv b_1$ in the vertical vorticity equation:

$$
\partial_t \zeta + (\psi, \zeta) = 0 
$$

$$
\partial_t \psi + (\psi, \psi) = 0 
$$

$$
\partial_t \delta \psi + (\psi, \delta \psi) = 0 
$$

• with quasigeostrophic potential vorticity $\psi$.

5 Linear Dispersion Relations

Using $\kappa = (\text{a} + \kappa^2 + \kappa^2) - k^2 U^2 \omega$, for $\kappa$, constant, dispersion relations for the 3D parent Boussinesq, the reduced rotation model, and the quasi-geostrophic equations are:

$$
\text{Boussinesq: } \omega^2 = \kappa^2 \rho_0 \beta / \rho_b, \text{ or } \omega = \kappa \sqrt{\beta / \rho_b} 
$$

$$
\text{rotation constrained: } \omega^2 = \kappa^2 \rho_0 \beta / \rho_b + \kappa^2 \rho_0 \beta / \rho_b, \text{ or } \omega = \kappa \sqrt{2 \beta / \rho_b} 
$$

6 Hamiltonian Formulation

• In the inviscid case, the Hamiltonian/energy of (2) is:

$$
H = \frac{1}{2} \int \left| \nabla \psi \right|^2 + \frac{1}{2} \int \left| \nabla \delta \psi \right|^2 + \frac{1}{2} \int \left| \nabla \delta \psi \right|^2 \mathrm{d}x \mathrm{d}y \mathrm{d}z 
$$

upon using the boundary conditions $\psi = 0$ at $r = R$ and $\psi = 0$ at $z = 0, H_T$.

• Variations of the Hamiltonian are:

$$
\delta H = \delta \int \left| \nabla \psi \right|^2 + \frac{1}{2} \int \left| \nabla \delta \psi \right|^2 \mathrm{d}x \mathrm{d}y \mathrm{d}z 
$$

$$
\delta H = \delta \int \left| \nabla \delta \psi \right|^2 + \frac{1}{2} \int \left| \nabla \delta \psi \right|^2 \mathrm{d}x \mathrm{d}y \mathrm{d}z 
$$

7 Numerical Weak Formulation

• Consequently, a weak formulation and candidate Hamiltonian formulation reads

$$
\mathcal{L} \Phi = \left( \int \frac{\partial F}{\partial \psi} \mathrm{d}x \mathrm{d}y \mathrm{d}z + \int \frac{\partial F}{\partial \delta \psi} \mathrm{d}x \mathrm{d}y \mathrm{d}z + \int \frac{\partial F}{\partial \delta \psi} \mathrm{d}x \mathrm{d}y \mathrm{d}z \right) 
$$

8 Phase Changes: Iodine Cycle

• Consider a container with dry air at room temperature and a small mass fraction of solid iodine particles on the bottom.

• Heat and keep the bottom above the iodine sublimation temperature $T_s = 183K$

• Keep the top below $T < T_s$ with a thermal surface repelling iodine solidification.

• Rotating Rayleigh-Bénard convection set-up.

• Total dimensional density is related to temperature as follows: $\rho = \rho(1 - \alpha T)$.

• A bulk two-state moisture model is adopted with iodine vapor $q$, and iodine snow/precipitate $q_s$, cf. similar approach in Zerroukat & Allen.

9 Future Work: Conceptual Laboratory Experiment & DA

References


