Iterative Kalman Filter using Ensembles

Technical formulation and preliminary test

Tannersvile PA / 8th Adjoint Workshop

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Vermelding onderdeel organisatie

Delft University of Technology

Outline

" Motivation
" Notation & review
" DUD
" Algorithm
" Experiment
" Summary

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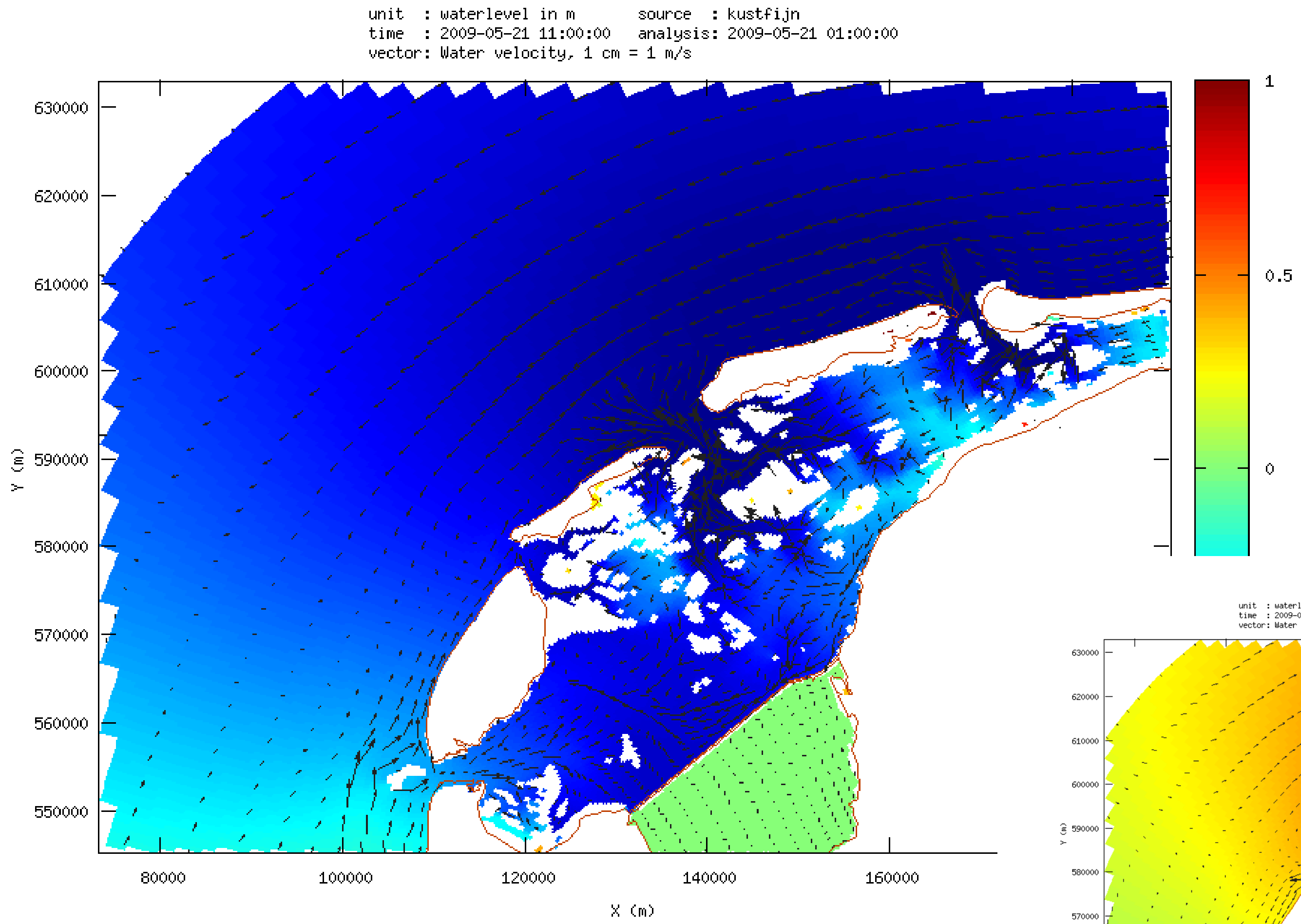
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vector: Water velocity, 1 cm = 1 m/s
630000
620000
610000
600000
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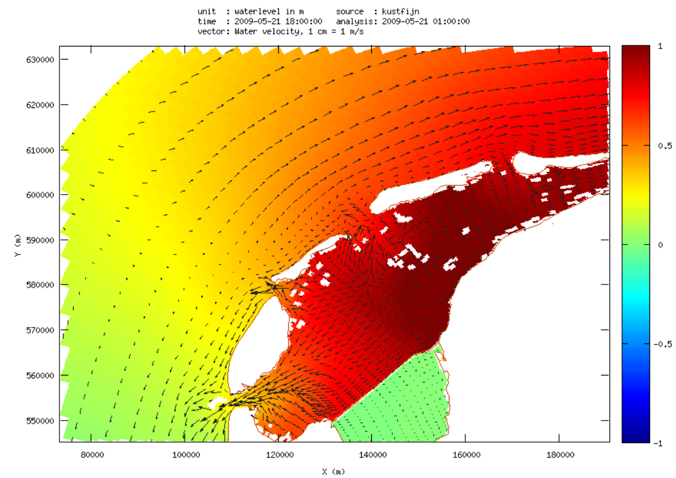
Y (m)
80000
100000
120000
140000
160000
X (m)
1
0.5
0


unit : waterlevel in m source : kustfijn
time : 2009-05-21 18:00:00 analysis: 2009-05-21 01:00:00
vector: Water velocity, 1 cm = 1 m/s
630000
620000
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Y (m)
80000
100000
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180000
X (m)
1
0.5
0
-0.5
-1

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Motivation

- particle filters have nice theoretical properties, but are very expensive to run
- Kalman Filter analysis is inconsistent with model if operators are non-linear (unlike VAR)

$$\mathbf{y}_k^a \neq H_k[\mathbf{x}_k^a]$$

$$\mathbf{x}_{k+1}^a \neq M_k[\mathbf{x}_k^a] + \mathbf{w}_k^a$$

$$\mathbf{x}_k^a \neq M_{0,k}[\mathbf{x}_0, \hat{\mathbf{p}}]$$

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Notation

$$\mathbf{x}_{k+1}^t = M_k[\mathbf{x}_k^t] + \mathbf{w}_k$$

$$\mathbf{y}_k^o = H_k[\mathbf{x}_k^t] + \varepsilon_k$$

$$\mathbf{J}(\mathbf{x}_k) = \frac{1}{2}(\mathbf{x}_k - \mathbf{x}_k^f)'(\mathbf{P}_k^f)^{-1}(\mathbf{x}_k - \mathbf{x}_k^f) + \frac{1}{2}(\mathbf{y}_k^o - H_k[\mathbf{x}_k])'\mathbf{R}_k^{-1}(\mathbf{y}_k^o - H_k[\mathbf{x}_k])'$$

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Incremental formulation

$$\mathbf{x} = \mathbf{x}^g + \delta\mathbf{x}$$

$$\mathbf{J}_{incr}(\delta\mathbf{x}) = \frac{1}{2}(\delta\mathbf{x} + \mathbf{x}^g - \mathbf{x}^f)'(\mathbf{P}_k^f)^{-1}(\delta\mathbf{x} + \mathbf{x}^g - \mathbf{x}^f) + \frac{1}{2}(\mathbf{y}^o - H[\mathbf{x}^g] - \mathbf{H}\delta\mathbf{x})'\mathbf{R}_k^{-1}(\mathbf{y}^o - H[\mathbf{x}^g] - \mathbf{H}\delta\mathbf{x})$$

$$\nabla\mathbf{J}_{incr}(\delta\mathbf{x}) = (\mathbf{P}^f)^{-1}(\delta\mathbf{x} + \mathbf{x}^g - \mathbf{x}^f) - \mathbf{H}'\mathbf{R}^{-1}(\mathbf{y}^o - H[\mathbf{x}^g] - \mathbf{H}\delta\mathbf{x})$$

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Incremental form and Kalman

$$\delta\mathbf{x} = \left((\mathbf{P}^f)^{-1} + \mathbf{H}'\mathbf{R}^{-1}\mathbf{H}\right)^{-1} \left((\mathbf{P}^f)^{-1}(\mathbf{x}^g - \mathbf{x}^f) - \mathbf{H}'\mathbf{R}^{-1}(\mathbf{y}^o - H[\mathbf{x}^g])\right)$$

$$\mathbf{x}^f = \mathbf{x}^g \longrightarrow \delta\mathbf{x} = \mathbf{K}(\mathbf{y}^o - H[\mathbf{x}^g])$$

$$\mathbf{K} = \mathbf{P}^f\mathbf{H}'\left(\mathbf{H}\mathbf{P}^f\mathbf{H}' + \mathbf{R}\right)^{-1}$$

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Ensemble Kalman Filter

$$\mathbf{x}^f \approx \overline{\xi}^f = \frac{1}{n}\sum_{i=1}^{n}\xi_i^f \qquad \mathbf{P}^f \approx \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \overline{\xi}^f)(\xi_i - \overline{\xi}^f)'$$

$$\mathbf{P}^f = \mathbf{A}^f(\mathbf{A}^f)'$$

$$\mathbf{A}^f = \left[\frac{1}{\sqrt{n-1}}(\xi_1^f - \overline{\xi}^f), \ldots, \frac{1}{\sqrt{n-1}}(\xi_n^f - \overline{\xi}^f)\right]$$

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Ensemble based VAR

$$\mathbf{x} = \overline{\xi}^f + A\mathbf{w}$$

$$\begin{aligned}\mathbf{J}_{incr}(\mathbf{w}) &= \frac{1}{2}(\mathbf{A}^f\mathbf{w})'(\mathbf{P}_k^f)^{-1}(\mathbf{A}^f\mathbf{w}) + \frac{1}{2}(\mathbf{y}^o - H[\overline{\xi}^f] - \mathbf{HA}^f\mathbf{w})'\mathbf{R}_k^{-1}(\mathbf{y}^o - H[\overline{\xi}^f] - \mathbf{HA}^f\mathbf{w}) \\ &= \frac{1}{2}\mathbf{w}'\mathbf{w} + \frac{1}{2}(\mathbf{y}^o - H[\overline{\xi}^f] - \mathbf{HA}^f\mathbf{w})'\mathbf{R}_k^{-1}(\mathbf{y}^o - H[\overline{\xi}^f] - \mathbf{HA}^f\mathbf{w})\end{aligned}$$

$$\mathbf{HA}^f \approx \left[\frac{1}{\sqrt{n-1}}(H\xi_1^f - H\overline{\xi}^f), \ldots, \frac{1}{\sqrt{n-1}}(H\xi_n^f - H\overline{\xi}^f)\right]$$

ETKF, ENSR, MLEF, ...
EnRML Gu & Oliver 2007 (SPE)
En3DVAR Liu et. al. 2008

DUD - Does not Use Derivatives

$$\mathbf{J}(\mathbf{w}) = \frac{1}{2}(\mathbf{y}^o - \mathbf{f}[\mathbf{w}])'(\mathbf{y}^o - \mathbf{f}[\mathbf{w}])$$

$$\mathbf{f}(\mathbf{w}) \approx \mathbf{f}(\mathbf{w}_0) + [\mathbf{f}(wb_1) - \mathbf{f}(wb_0), \ldots, \mathbf{f}(wb_q) - \mathbf{f}(wb_0)] \\ [wb_1 - wb_0, \ldots, wb_q - wb_0]^{-1}(\mathbf{w} - \mathbf{w}_0)$$

Ralston & Jennrich 1978

DUD solve

$$\mathbf{A} = [\mathbf{f}(\mathbf{w}_1) - \mathbf{f}(\mathbf{w}_0), \ldots, \mathbf{f}(\mathbf{w}_q) - \mathbf{f}(\mathbf{w}_0)]$$

$$\mathbf{B} = [\mathbf{w}_1 - \mathbf{w}_0, \ldots, \mathbf{w}_q - \mathbf{w}_0]$$

$$\mathbf{J}_{lin}(\mathbf{w}) = \frac{1}{2}\left(\mathbf{y}^o - \mathbf{f}(\mathbf{w}_0) + \mathbf{B}\mathbf{A}^{-1}(\mathbf{w} - \mathbf{w}_0)\right)'\left(\mathbf{y}^o - \mathbf{f}(\mathbf{w}_0) + \mathbf{B}\mathbf{A}^{-1}(\mathbf{w} - \mathbf{w}_0)\right)$$

$$\hat{\mathbf{w}} = argmin_{\mathbf{w}}\mathbf{J}_{lin}(\mathbf{w})$$

Now update trial points $\mathbf{w}_j$

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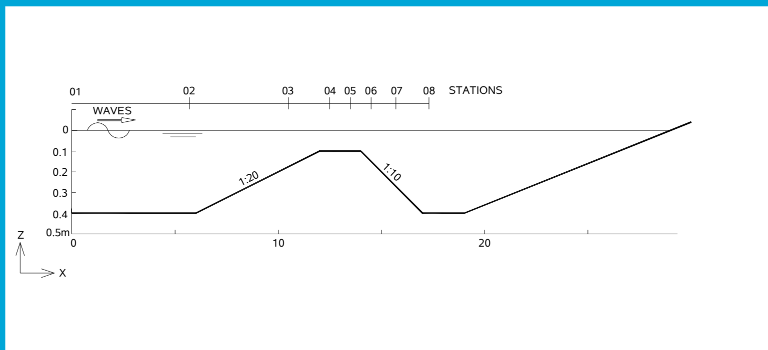
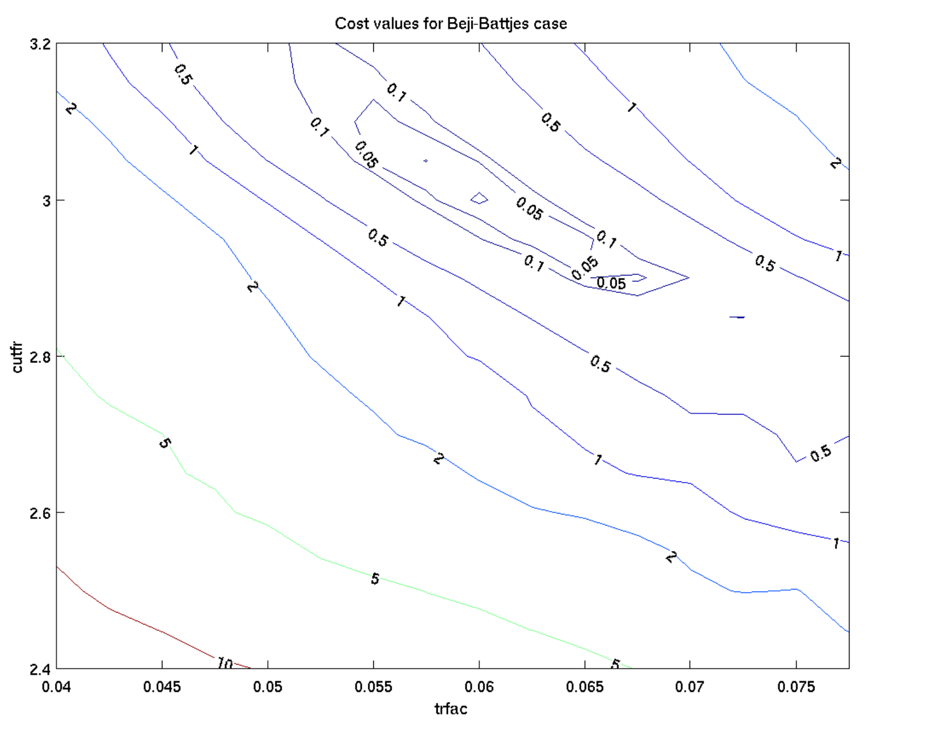
DUD example



$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial x}\left(c_x N\right)+\frac{\partial}{\partial y}\left(c_y N\right)+\frac{\partial}{\partial \sigma}\left(c_\sigma N\right)+\frac{\partial}{\partial \theta}\left(c_\theta N\right)=\frac{S_{tot}}{\sigma}$$



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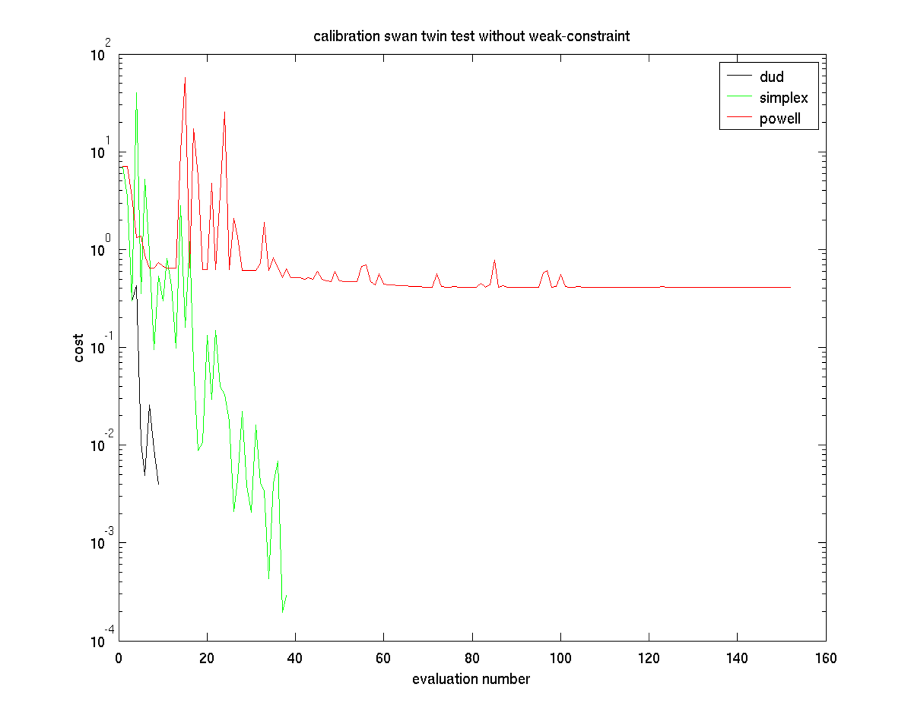
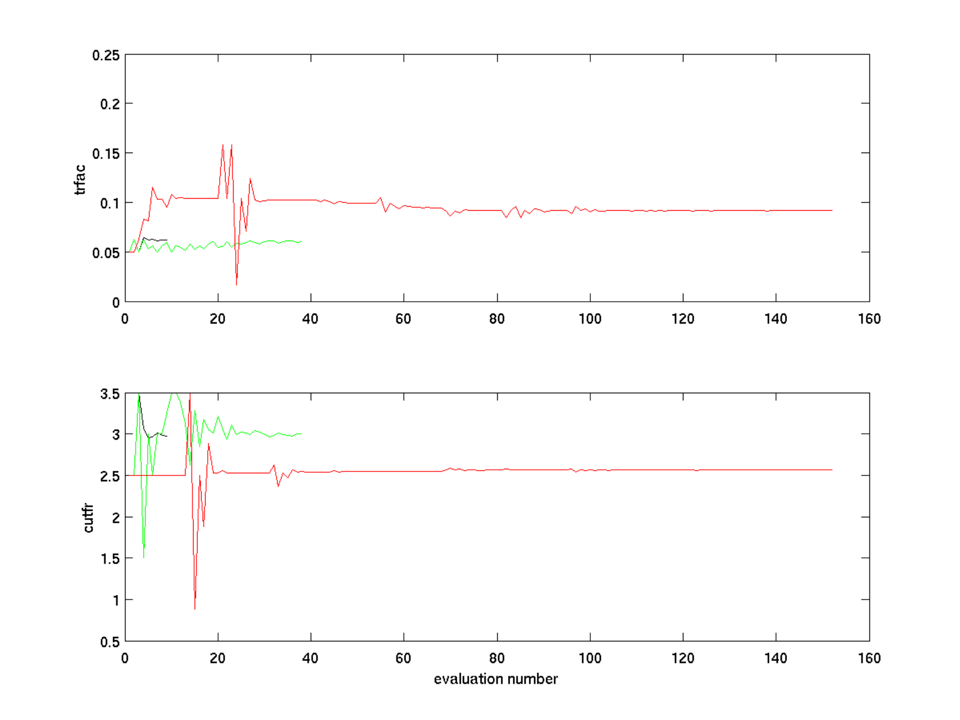
DUD example





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DUDEnKF

substititute $\mathbf{x} = \overline{\xi}^f + A\mathbf{w}$ in non-linear $\mathbf{J}(\mathbf{x}_k)$

$$\mathbf{J}(\mathbf{w}) = \frac{1}{2}\mathbf{w}'\mathbf{w} + \frac{1}{2}(\mathbf{y}^o - H[\overline{\xi}^f + A\mathbf{w}])'\mathbf{R}^{-1}(\mathbf{y}^o - H[\overline{\xi}^f + A\mathbf{w}])$$

$$H_k[\overline{\xi}^f + A\mathbf{w}] \approx H[\overline{\xi}^f + A\mathbf{w}_1^l]$$

$$+\left[H[\overline{\xi}^f + A\mathbf{w}_2^l] - H[\overline{\xi}^f + A\mathbf{w}_1^l], \ldots, H[\overline{\xi}^f + A\mathbf{w}_n^l] - H[\overline{\xi}^f + A\mathbf{w}_1^l]\right]$$

$$\left[\mathbf{w}_2^l - \mathbf{w}_1^l, \ldots, \mathbf{w}_n^l - \mathbf{w}_1^l\right]^{-1}(\mathbf{w} - \mathbf{w}_1^l)$$

MLEF Zupanski 2005

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DUDEnKF algorithm

Time loop

" forecast $\xi^f_{k+1} = M_k[\xi^a{}_i] + \mathbf{w}_{k\,i}$

" minimize $\mathbf{J}_i(\mathbf{w}) = \frac{1}{2}\mathbf{w}'\mathbf{w}$

$$+\frac{1}{2}(\mathbf{y}^o + \varepsilon_{k\,i} - H[\xi^f_i + A\mathbf{w}])'\mathbf{R}^{-1}(\mathbf{y}^o + \varepsilon_{k\,i} - H[\bar{\xi}^f + A\mathbf{w}])$$

using DUD, $\xi^a{}_i = \bar{\xi}^f + A\hat{\mathbf{w}}_i$

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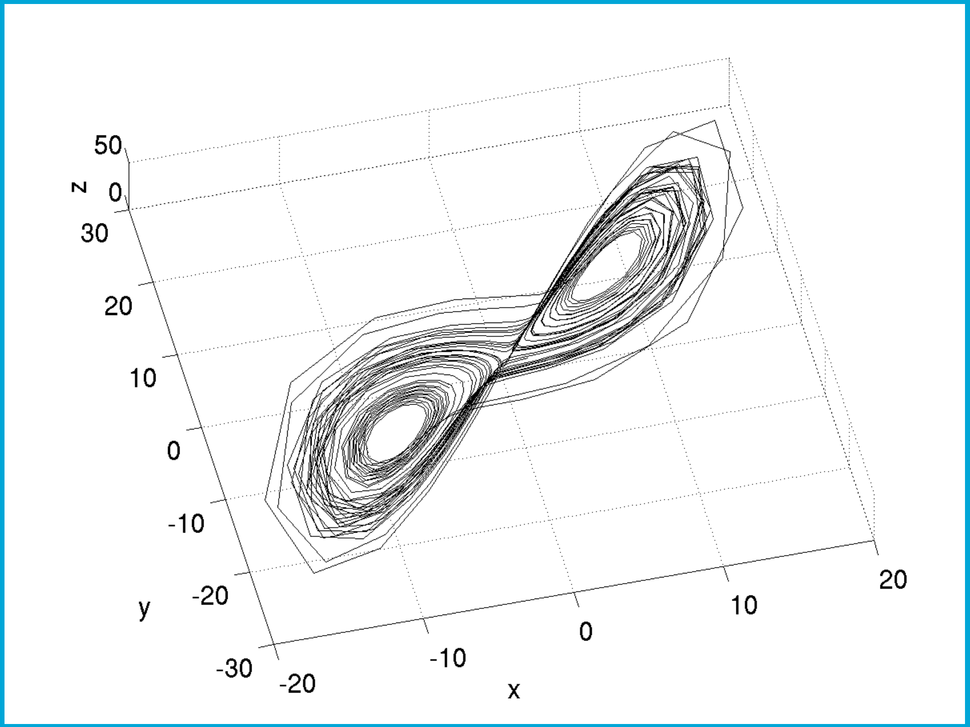
Lorenz example

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = xy - \beta z$$

observations x^2, y^2, z^2

$$\Delta t_{obs} = 0.2$$

$$\sigma_{obs} = 10.0$$



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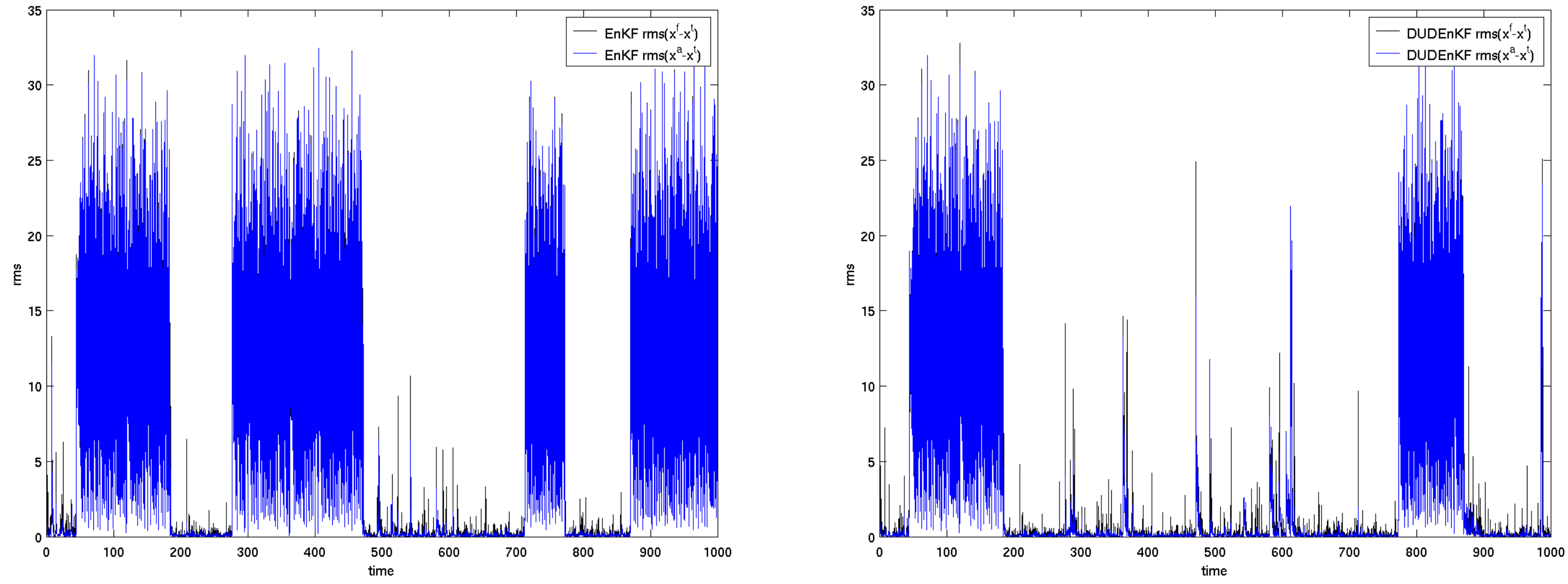
Lorenz example



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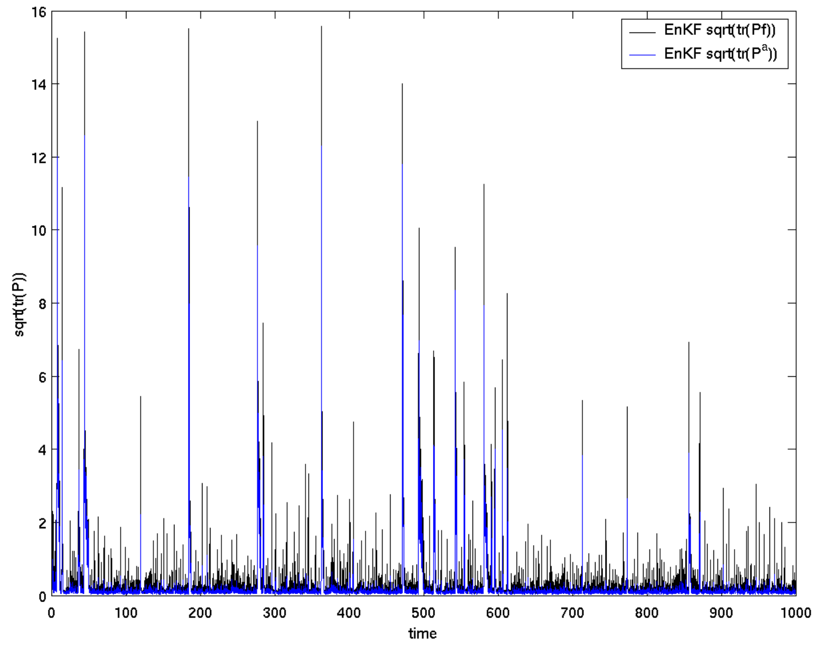
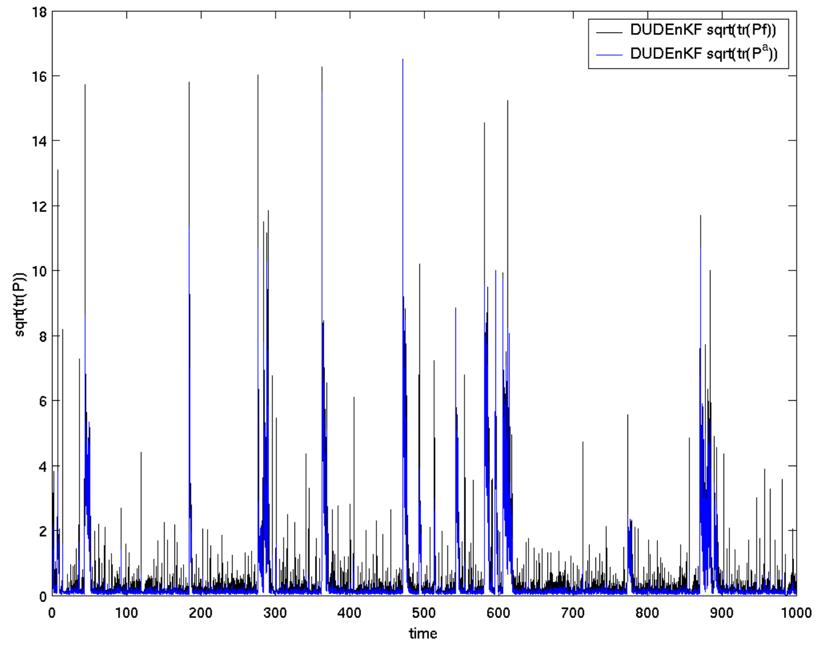
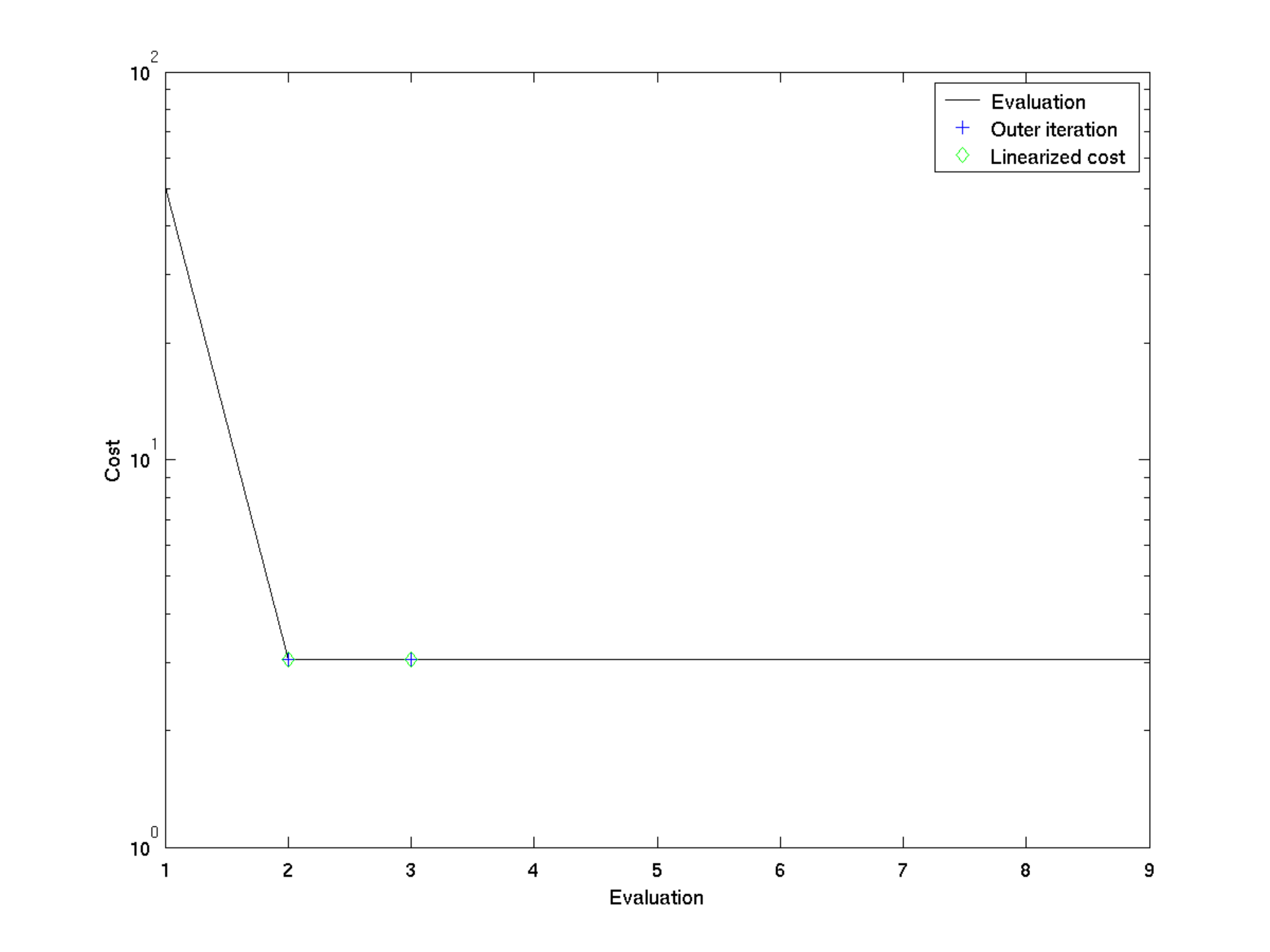
Lorenz example

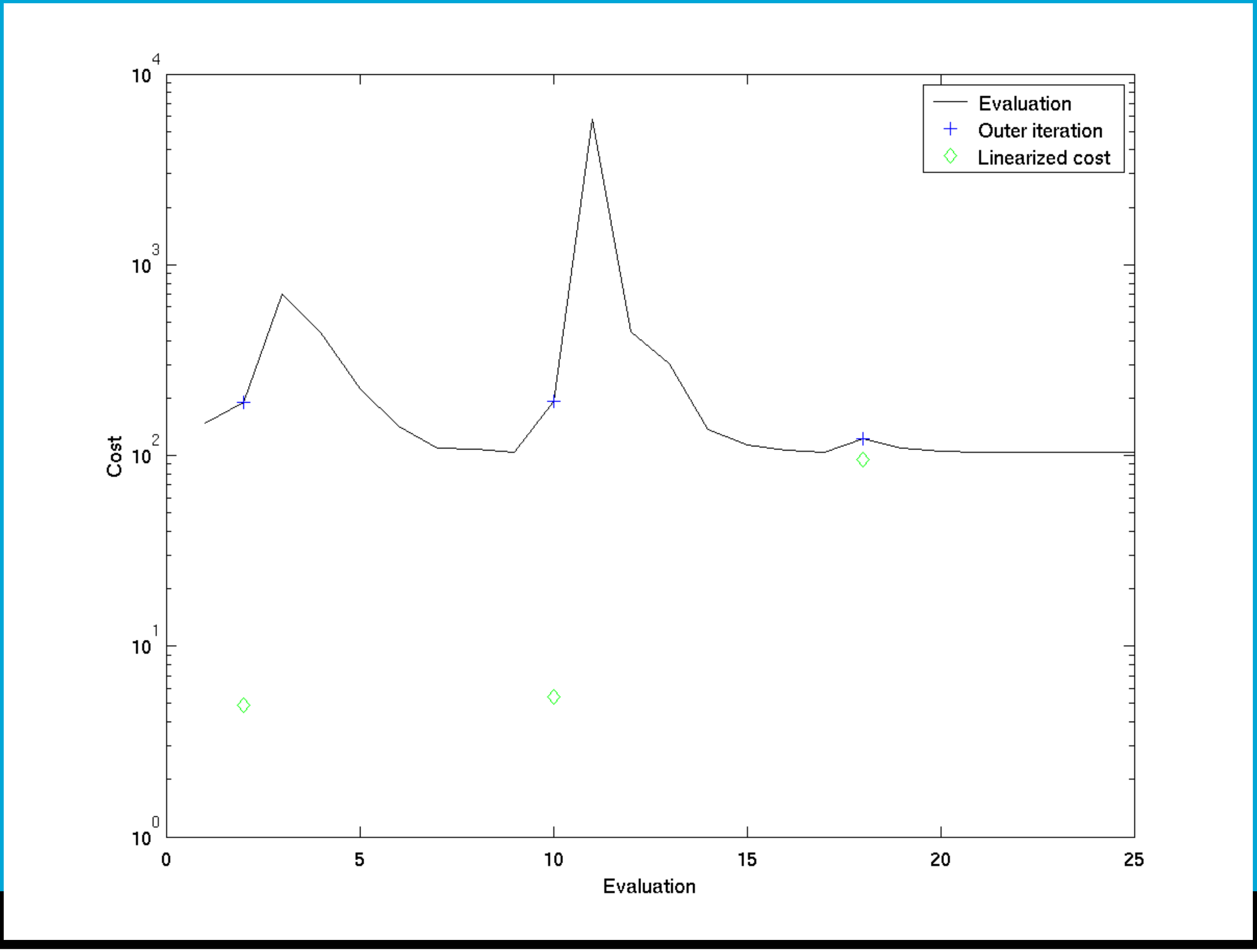




Iteration – quick convergence



Iteration – slow convergence



Summary / future

" Iterations in a maximum-likelihood/VAR approach can improve consistency
" Iterated Kalman Filter can approach VAR solution
" No TLM or adjoint needed (except for efficiency)

" Next: smoothing and parameter estimation and applications

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