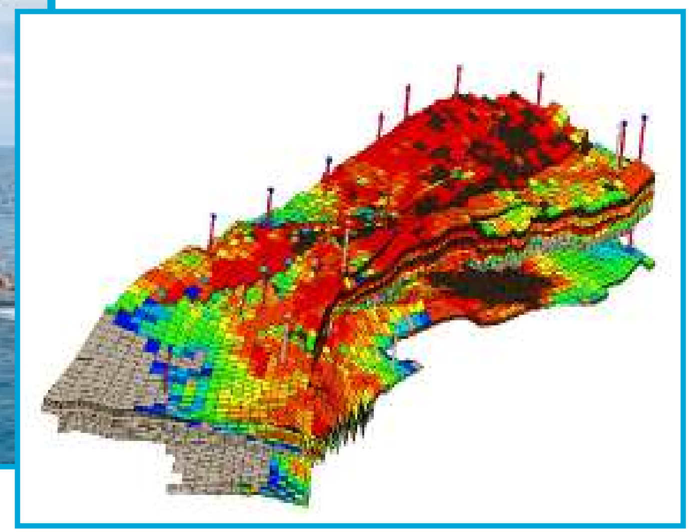
Model-Reduced Variational Data Assimilation For Reservoir Model Updating

Delft University of Technology



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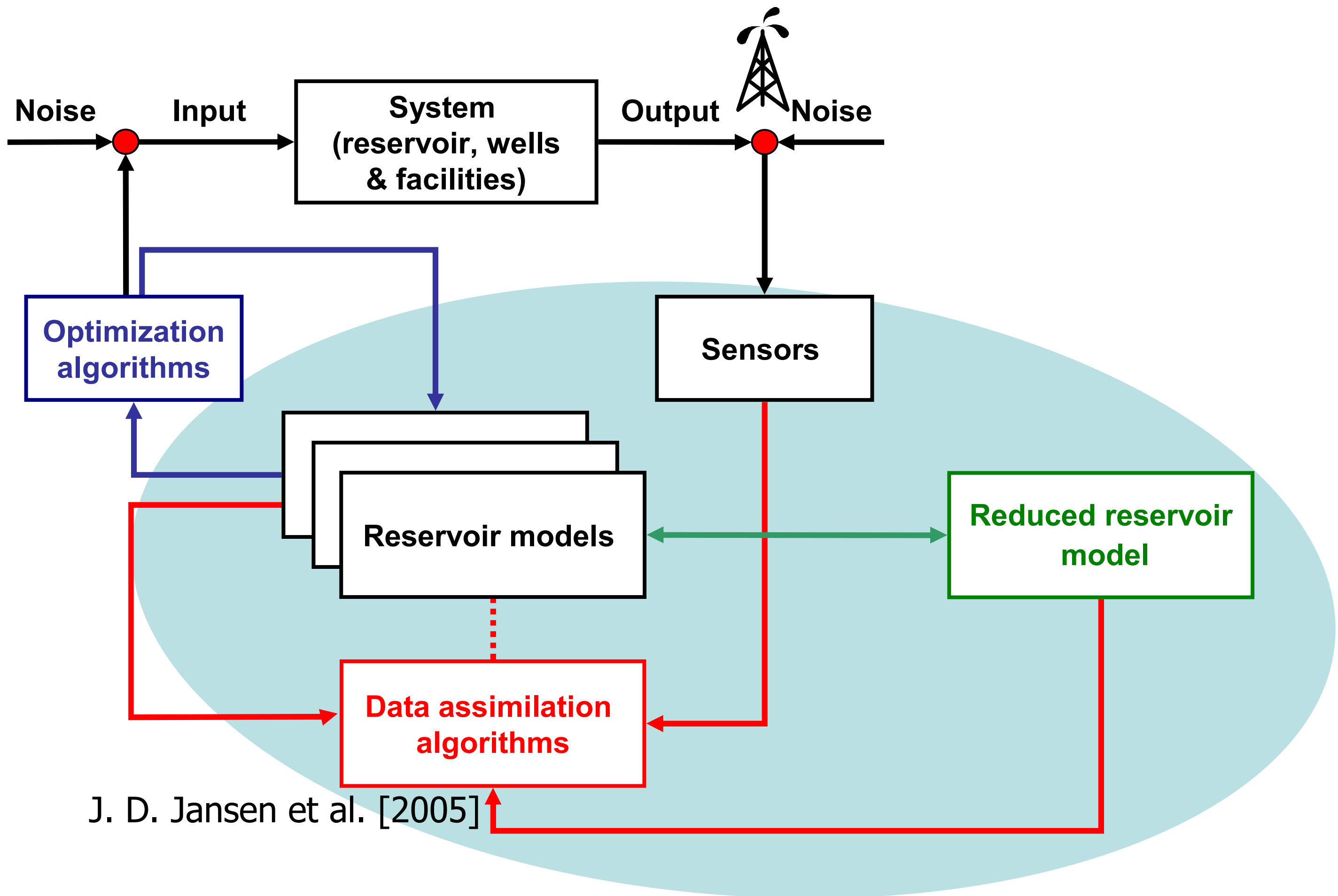
Outline

- Background
- Model-reduced variational data assimilation (MRVDA)
- Study case
- Results
- Conclusions

Background

Reservoir management represented as a model-based closed-loop controlled process



J. D. Jansen et al. [2005]

Background

Goal

To find the **maximum a posteriori estimate** of the log permeability field by solving the minimization problem:

$$\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) = \frac{1}{2}\left(\boldsymbol{\theta}-\boldsymbol{\theta}^{prior}\right)^T \mathbf{P}_{\theta}^{-1}\left(\boldsymbol{\theta}-\boldsymbol{\theta}^{prior}\right) + \frac{1}{2}\sum_{i=1}^{N_D}\left(\mathbf{y}_i^{obs}-\mathbf{y}_i(\boldsymbol{\theta})\right)^T \mathbf{P}_i^{-1}\left(\mathbf{y}_i^{obs}-\mathbf{y}_i(\boldsymbol{\theta})\right)$$

- $\mathbf{y}_i^{obs}, \mathbf{y}_i$ represent the vectors of observed and predicted production data at time t_i,
- $\boldsymbol{\theta}$ the uncertain log permeability vector,
- $\boldsymbol{\theta}^{prior}$ the prior knowledge about log permeability,
- $\mathbf{P}_{\theta}$ covariance matrix for the log permeability vector,
- $\mathbf{P}_i$ the covariance matrix for data measurements errors

MRVDA

Advantage of adjoint method

- A gradient based algorithm with gradient calculated by adjoint method which requires one adjoint solution regardless of the number of model parameters

Disadvantage of adjoint method

- Implementation of the adjoint model

Solutions

If we re-parameterize the permeability field then:

- we can calculate the gradient by perturbations: computationally expensive for large number of uncertain parameters

- we can construct the low-order approximation of the tangent linear reservoir model for which the low-order adjoint model is derived

MRVDA

Proper Orthogonal Decomposition (Lumley 1967) also knows as

- Karhunen- Loève (K-L) Transform
  - Loève (1946)
  - Karhunen (1946)
- Empirical Orthogonal Function
- Principal Component Analysis (1986)

Application

- Statistical tool to analyze experimental data

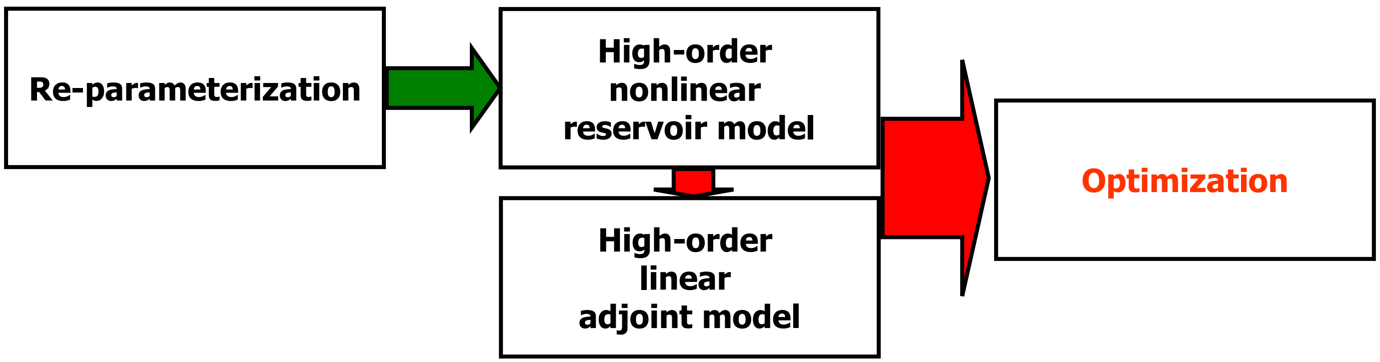
  The POD is used to analyze the set of realizations with an aim to extract dominant features and trends (coherent structures called *patterns* in space)

- Reduced Order Modeling (ROM)

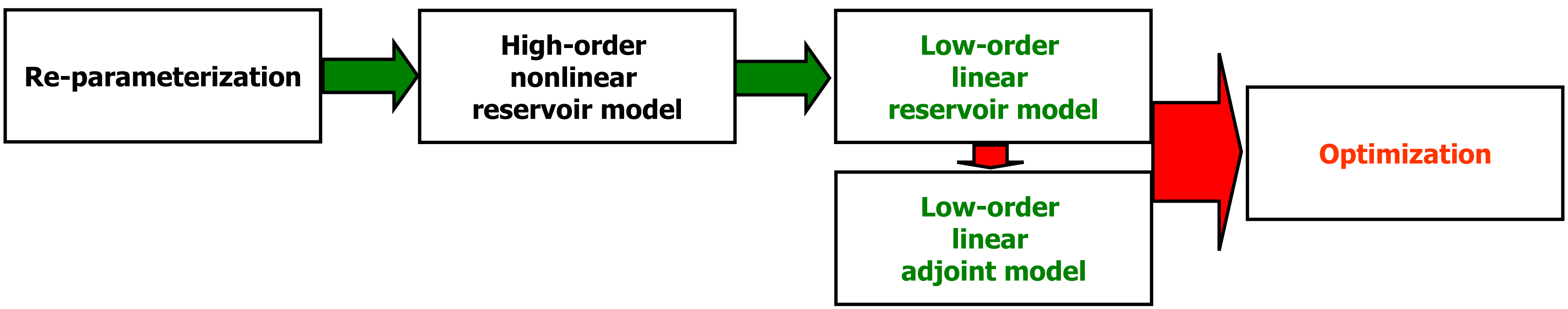
  The POD is used to provide a relevant set of *basis functions* with which a low-dimensional subspace is identified, then the reduced model is constructed by projection of the governing equations on that subspace

MRVDA

Classical adjoint approach



MRVDA



Vermeulen, P.T.M., Heemink, A.W. [2006]
Altaf, U.M., Inverse shallow water flow modeling

MRVDA

High-order nonlinear model ➡ **high-order linearized model**

Consider high-order **nonlinear** reservoir model

$$\mathbf{x}(t_i) = \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right) \in \mathbf{R}^h, \quad h \sim O(10^6) \quad h_\theta = |\boldsymbol{\theta}| \sim O(10^6)$$

Tangent **linear** approximation of the high-order nonlinear reservoir model around $(\mathbf{x}^b(t_i), \boldsymbol{\theta}^b)$ can be rewritten as

$$\Delta\mathbf{x}(t_i) = \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right)}{\partial \mathbf{x}(t_{i-1})} \Delta\mathbf{x}(t_{i-1}) + \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}} \Delta\boldsymbol{\theta}$$

where

$$\Delta\boldsymbol{\theta} = \boldsymbol{\theta} - \boldsymbol{\theta}^b \qquad \text{and} \qquad \Delta\mathbf{x}(t_i) = \mathbf{x}(t_i) - \mathbf{x}^b(t_i)$$

Step 0

Re-parameterize permeability field using set of realizations $\boldsymbol{\Theta} = \left\{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{K_\theta}\right\}$

$$\Delta\boldsymbol{\theta} = \mathbf{P}^\theta \mathbf{r}^\theta, \quad \mathbf{r}^\theta \in \mathbf{R}^{l_\theta}, l_\theta \ll h_\theta$$

MRVDA

High-order linearized model ➡ **low-order model**

Step 1

Generate number of system solutions and define *snapshots* as

$$\Delta\mathbf{x}(t_i) = \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}^b + \mathbf{P}^\theta \mathbf{r}^\theta\right) - \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}), \boldsymbol{\theta}^b\right)$$

Step 2

Apply POD on pressure and saturation snapshots separately and use it to derive projection subspace

$$\Delta\mathbf{x}(t_i) \approx \mathbf{P}\mathbf{r}(t_i) \qquad \mathbf{r} \in \mathbf{R}^l, l \ll h$$

Project the **high-order** linearized model

$$\Delta\mathbf{x}(t_i) = \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right)}{\partial \mathbf{x}(t_{i-1})} \Delta\mathbf{x}(t_{i-1}) + \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}} \Delta\boldsymbol{\theta}$$

into **low-order** linear model

$$\mathbf{r}(t_i) = \mathbf{N}_i \mathbf{r}(t_{i-1}) + \mathbf{N}_i^\theta \mathbf{r}^\theta$$

MRVDA

Step 3

Approximate the matrices of the low-order model

$$\mathbf{N}_i = \mathbf{P}^T \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}),\boldsymbol{\theta}\right)}{\partial \mathbf{x}(t_{i-1})}\mathbf{P} \in \mathbf{R}^{l\times l} \qquad \mathbf{N}_i^{\theta} = \mathbf{P}^T \frac{\partial \mathbf{f}_i\left(\mathbf{x}(t_{i-1}),\boldsymbol{\theta}\right)}{\partial \boldsymbol{\theta}}\mathbf{P}^{\theta} \in \mathbf{R}^{l\times l_{\theta}}$$

Using the chain rule differentiation

$$\frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1})+\mathbf{P}\mathbf{r}(t_{i-1}),\boldsymbol{\theta}^b\right)}{\partial \mathbf{r}_j(t_{i-1})} = \frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1})+\mathbf{P}\mathbf{r}(t_{i-1}),\boldsymbol{\theta}^b\right)}{\partial \mathbf{x}(t_{i-1})}\frac{\partial \mathbf{x}(t_{i-1})}{\partial \mathbf{r}_j(t_{i-1})} = \frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1})+\mathbf{P}\mathbf{r}(t_{i-1}),\boldsymbol{\theta}^b\right)}{\partial \mathbf{x}(t_{i-1})}\mathbf{P}_j$$

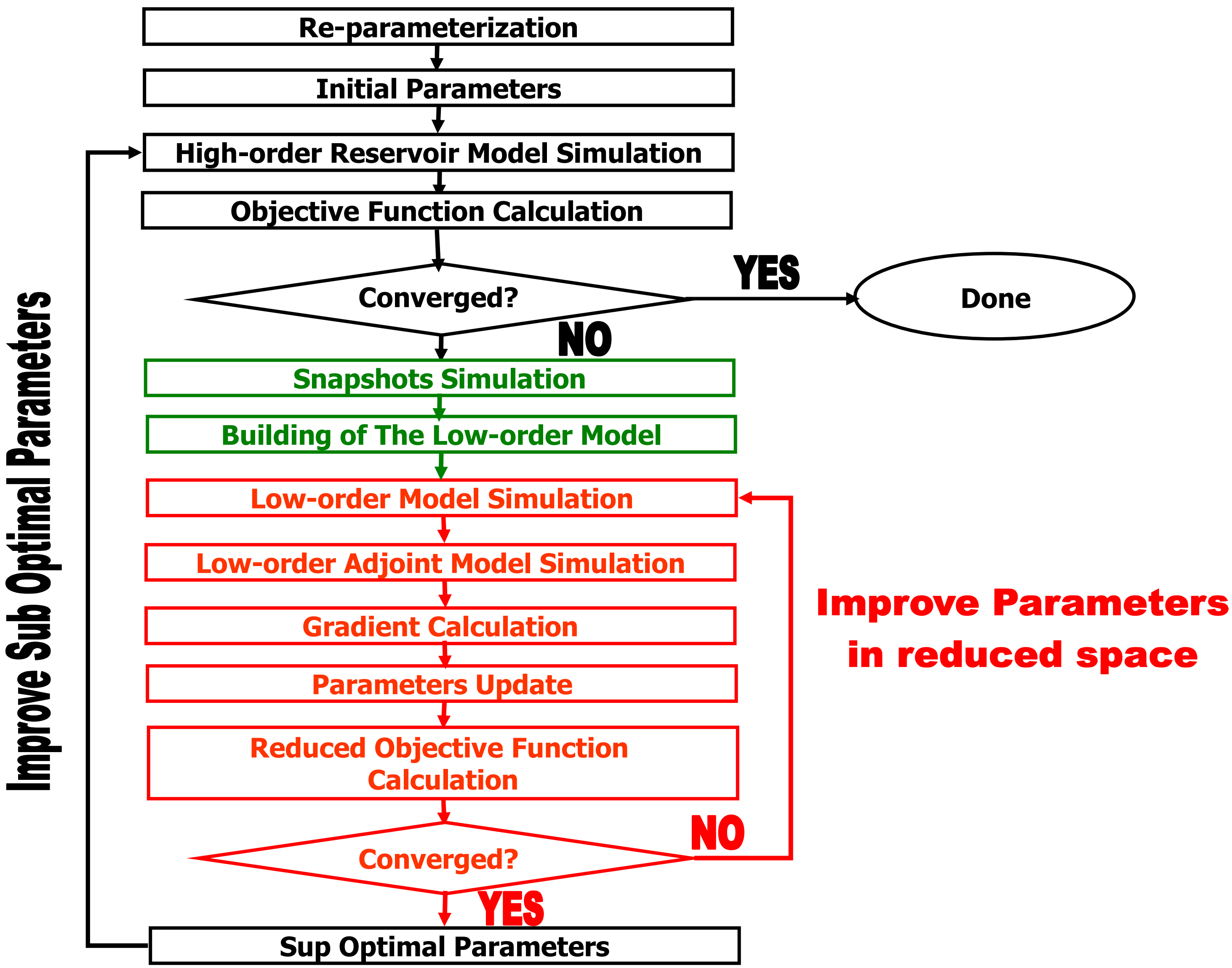
$$\frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b+\mathbf{P}^{\theta}\mathbf{r}^{\theta}\right)}{\partial \mathbf{r}_j^{\theta}} = \frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b+\mathbf{P}^{\theta}\mathbf{r}^{\theta}\right)}{\partial \boldsymbol{\theta}}\frac{\partial \boldsymbol{\theta}}{\partial \mathbf{r}_j^{\theta}} = \frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b+\mathbf{P}^{\theta}\mathbf{r}^{\theta}\right)}{\partial \mathbf{r}_j^{\theta}}\mathbf{P}_j^{\theta}$$

Using **finite difference approximation** we get

$$\frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b\right)}{\partial \mathbf{x}(t_{i-1})}\mathbf{P}_j \approx \frac{\mathbf{f}_i\left(\mathbf{x}^b(t_{i-1})+\varepsilon\mathbf{P}_j,\boldsymbol{\theta}^b\right)-\mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b\right)}{\varepsilon}$$

$$\frac{\partial \mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b\right)}{\partial \boldsymbol{\theta}}\mathbf{P}_j^{\theta} \approx \frac{\mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b+\varepsilon^{\theta}\mathbf{P}_j^{\theta}\right)-\mathbf{f}_i\left(\mathbf{x}^b(t_{i-1}),\boldsymbol{\theta}^b\right)}{\varepsilon^{\theta}}$$

MRVDA



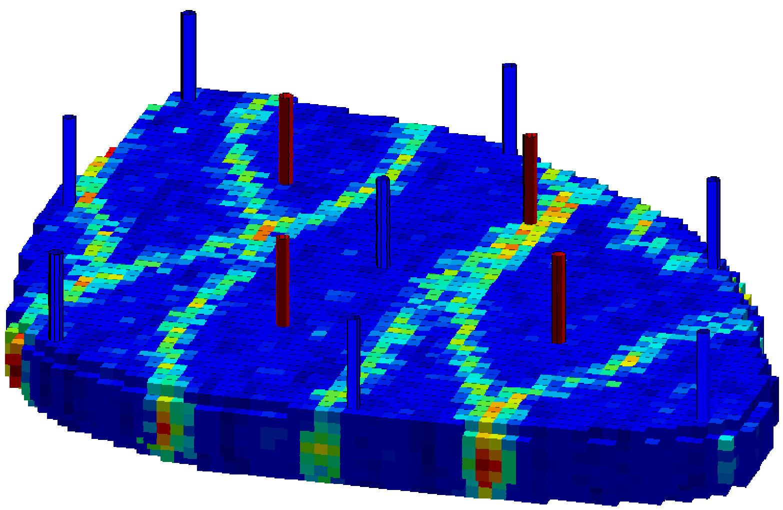
Re-parameterization
Initial Parameters
High-order Reservoir Model Simulation
Objective Function Calculation
Converged?
YES
Done
NO
Snapshots Simulation
Building of The Low-order Model
Low-order Model Simulation
Low-order Adjoint Model Simulation
Gradient Calculation
Parameters Update
Reduced Objective Function Calculation
Converged?
NO
YES
Sup Optimal Parameters
Improve Sub Optimal Parameters
Improve Parameters in reduced space

MRVDA

Computational complexity

- Preprocessing cost of generating representative snapshots spanning a large portion of possible permeability field
- Preprocessing cost of solving eigenvalue problems
- Preprocessing cost of building low-order system matrices
  - As many model runs + multiplications of Jacobians with projection matrix as many state patterns
  - As many model runs + multiplications of Jacobians with projection matrix as many parameters
- The cost of solving dense low-order linear model

Study case



Gijs van Essen [2006]

Producer

Injector

Reservoir model assumption

- 3 dimensional (60x60x7 with 18553 active grid blocks)
- Two-phase (oil-water)
- No-flow boundaries at all sides

Measurements

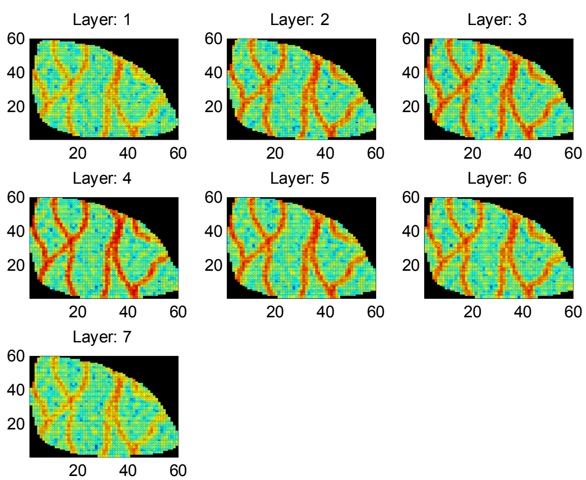
- Bottom hole pressures from injectors each 60 days during 3 years
- Flow rates from producers each 60 days during 3 years

Results

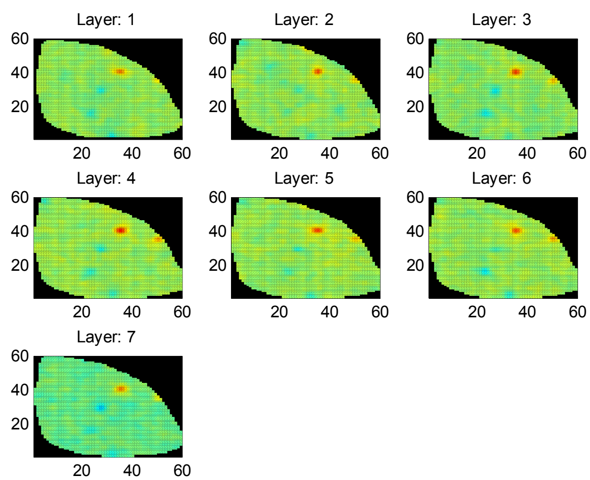
Model-Reduced VDA

Outer loop	Nr of model simulations	Objective function	Permeability patterns	State patterns	Number of snapshots
0	-	270	22	-	-
1	~ 67	130	22	29+5	200

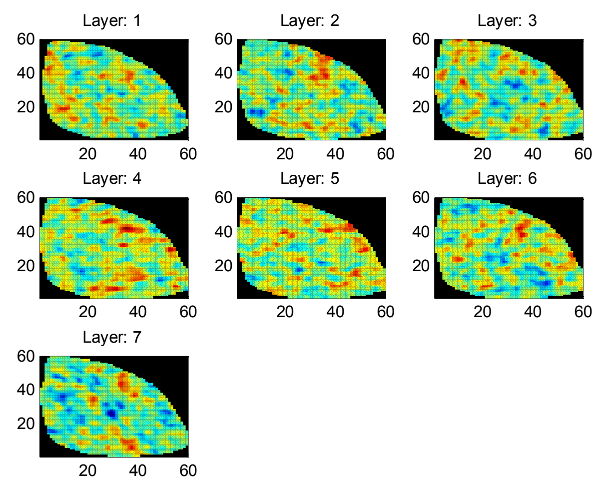
True log perm field



Prior log perm field

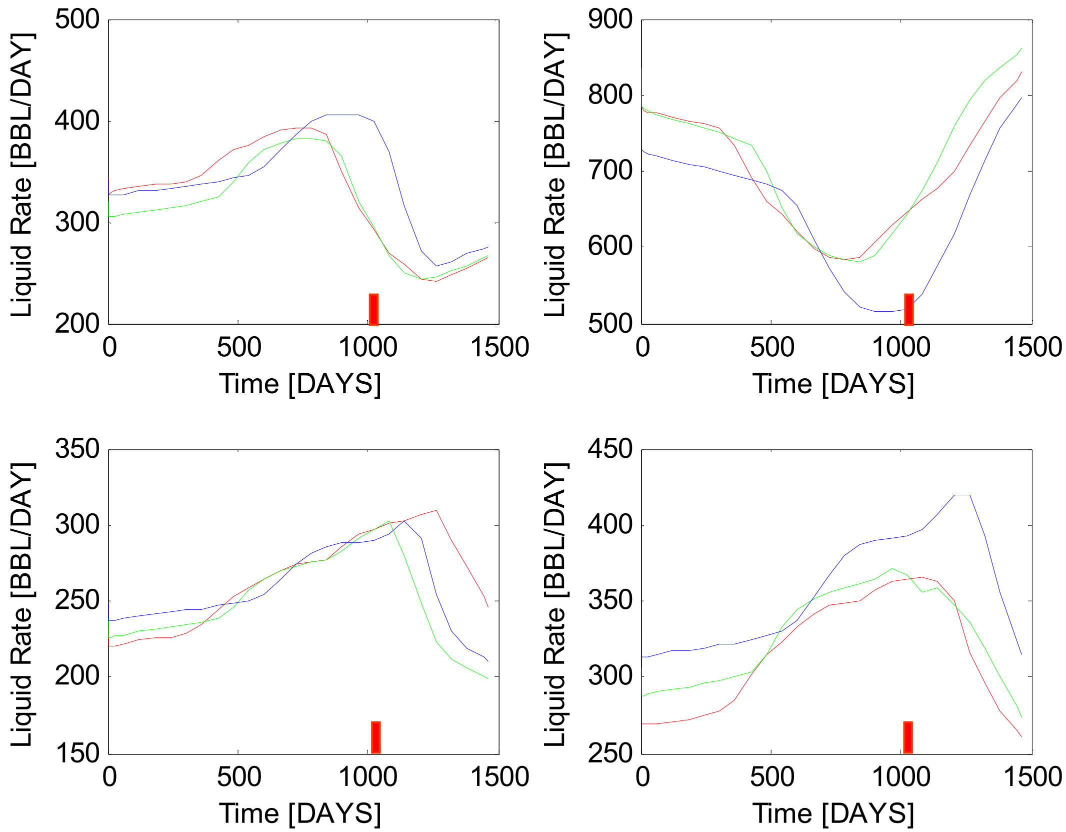


Estimated log perm field



Results

Liquid rate in the
production wells

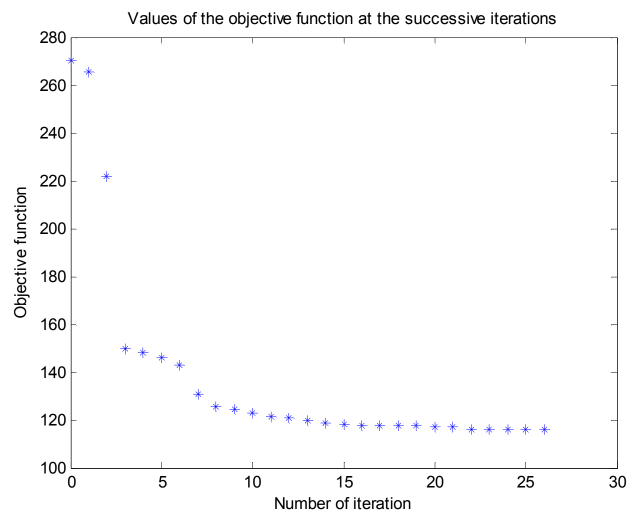


Legend

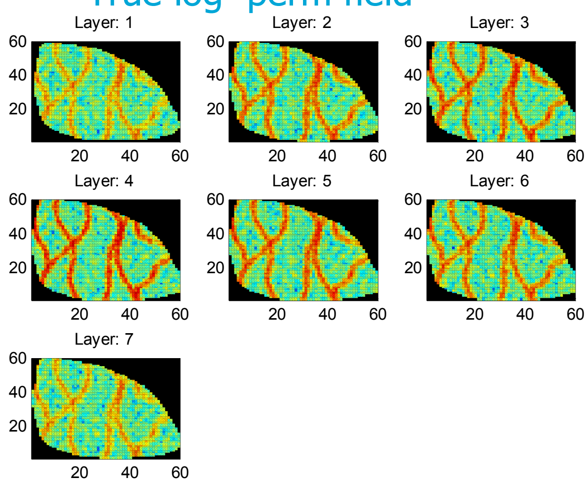
- true
- prior
- model-reduced

Results

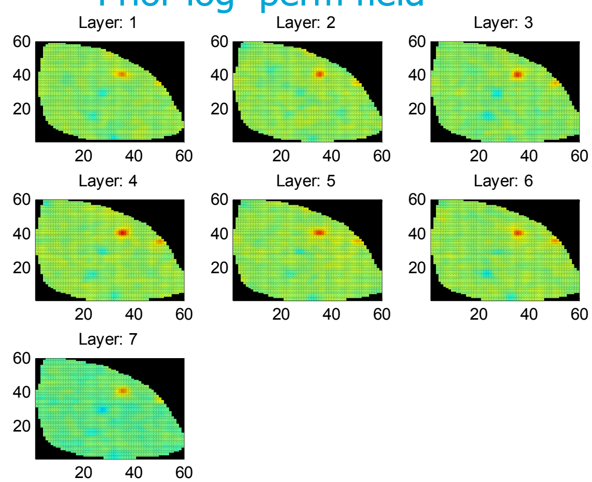
Classical adjoint approach



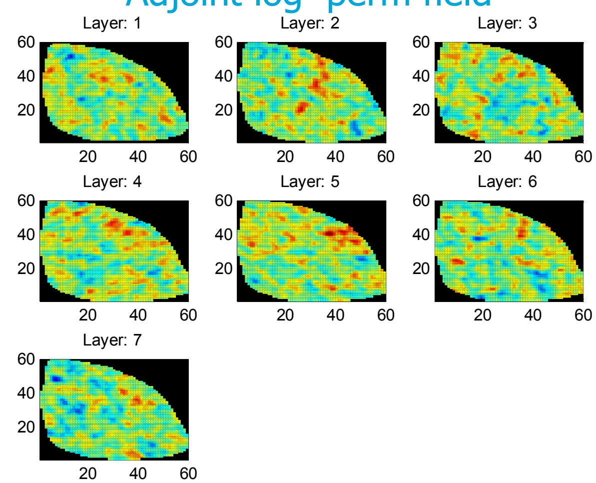
True log perm field



Prior log perm field

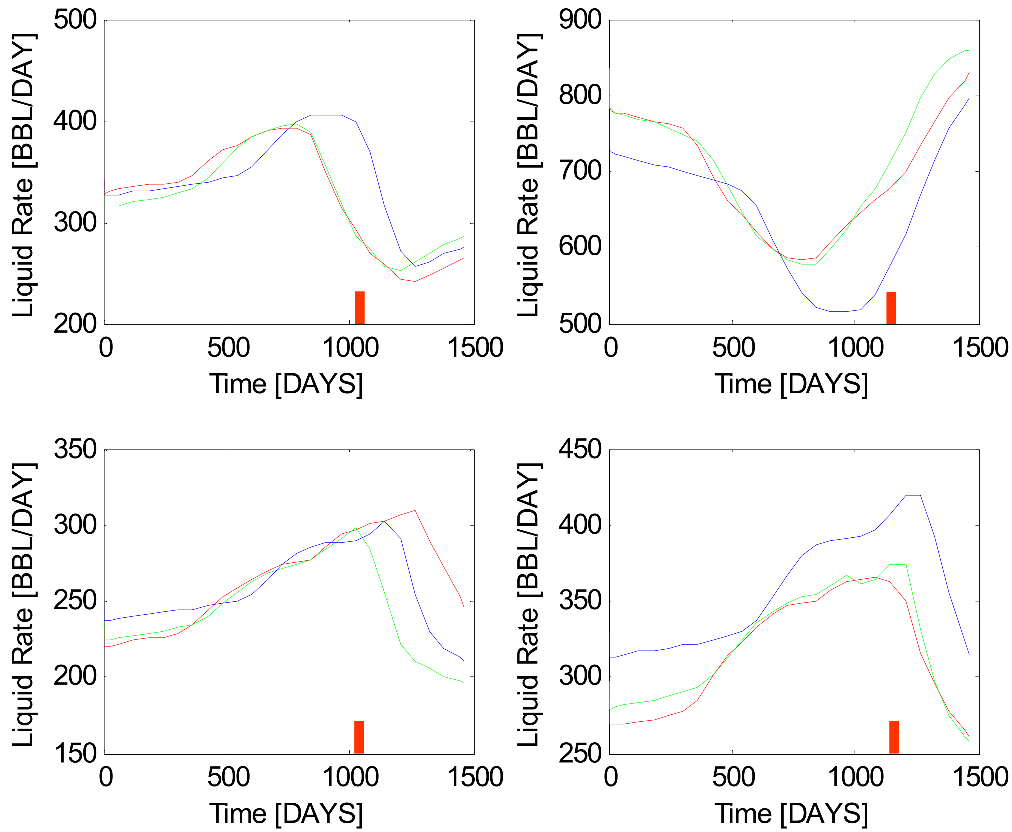


Adjoint log perm field

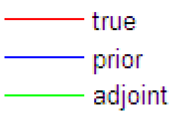


Results

Liquid rate in the production wells



Legend

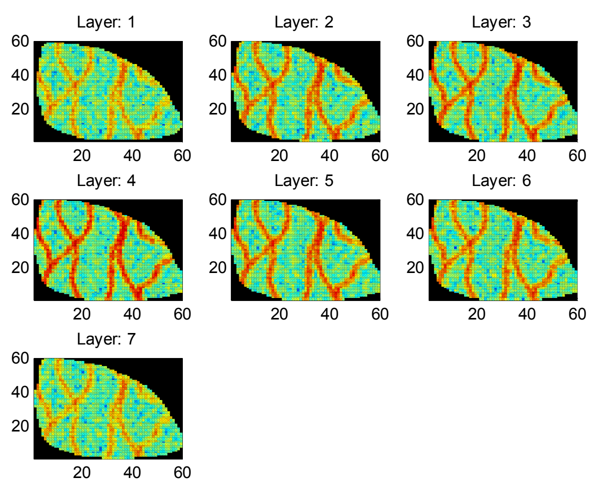


Results

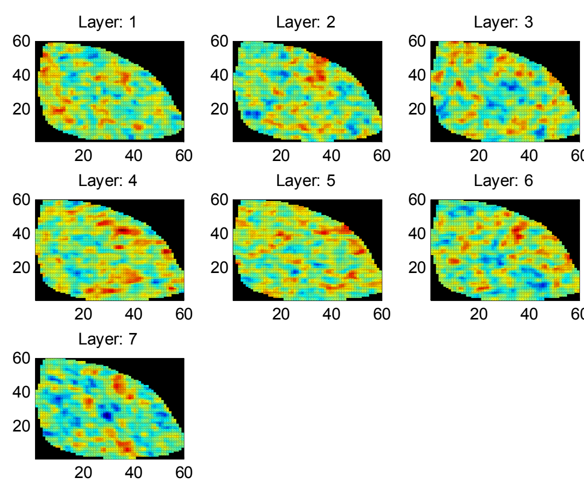
Comparison of the methods

Method	Objective function	Time in simulations
Initial (Prior)	270	-
Model-reduced approach	130	~67
Adjoint approach	116	~26*2

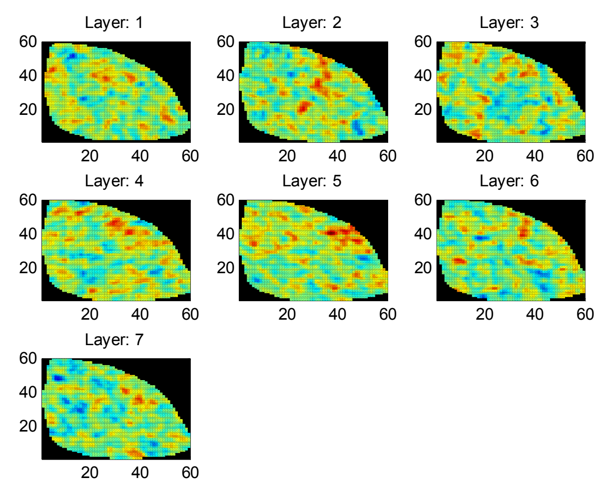
True log perm field



Model-reduced log perm field



Adjoint log perm field



Conclusions

- Model-reduced variational data assimilation does not require the implementation of the adjoint of the tangent linear model of the original reservoir model
- Model-reduced variational data assimilation gives the estimates comparable to those from data assimilation using an adjoint method (comparable match and predictions)
- Model-reduced variational data assimilation is easy to implement and treats simulator as a black-box
- If we have the adjoint code then we do go for the classical approach

Questions?

MRVDA: Proper Orthogonal Decomposition

Suppose we have an ensemble

$$\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_K\} \quad \mathbf{x}_i \in \mathbf{R}^h, \quad h \sim O(10^6)$$

POD seeks for *the best / optimal* linear representations of the members of the set $\mathbf{X}$:

$$\mathbf{X} \approx \hat{\mathbf{X}} = [\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_K]_{h\times K} = [\mathbf{p}_1, \ldots, \mathbf{p}_l]_{h\times l} \begin{bmatrix} r_1^1 & \cdots & r_K^1 \\ \vdots & \ddots & \vdots \\ r_1^l & \cdots & r_K^l \end{bmatrix}_{l\times K} \qquad \mathbf{p}_i \in \mathbf{R}^h$$

Optimality condition

$$\min_{\mathbf{P} \in R^{h\times l}} \frac{1}{K} \sum_{i=1}^{K} \left\| \mathbf{x}_i - \mathbf{P}\mathbf{P}^T \mathbf{x}_i \right\|_2$$

Solution

The above optimization problem is solved by eigenvalue analysis of the correlation matrix $\mathbf{R} = \mathbf{X}\mathbf{X}^T$, that is

$$\mathbf{X}\mathbf{X}^T \mathbf{p}_i = \lambda_i \mathbf{p}_i$$

MRVDA: Proper Orthogonal Decomposition

An optimal basis is given by

$$\mathbf{P}^* = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_l]$$

where $\mathbf{p}_i$ is the eigenvector corresponding to *i*-th largest eigenvalue $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l \gg \ldots \gg \lambda_K \geq 0$ of the matrix

The relative importance present in each basis vector

$$\varphi_j = \frac{\lambda_j}{\sum_{i=1}^{K} \lambda_i}$$

Choose l such that

$$\sum_{i=1}^{l} \varphi_i \leq \alpha$$

where α denotes the fraction of the cumulative relative importance we want to capture

MRVDA: Reduced Order Modeling

Consider a general high-order nonlinear discrete reservoir system

$$\mathbf{x}(t_i) = \mathbf{f}_i\left(\mathbf{x}(t_{i-1}), \boldsymbol{\theta}\right) \in \mathbf{R}^h,$$
$$\mathbf{y}(t_i) = \mathbf{h}_i\left(\mathbf{x}(t_i), \boldsymbol{\theta}\right)$$

The aim of the reduced order modeling is to find the projection $\mathbf{P} \in \mathbf{R}^{h \times l}$ with $\mathbf{P}^T\mathbf{P} = \mathbf{I}_l$ and $h \sim O(10^6)$ where $l \ll h$ to obtain the low-order system

$$\mathbf{r}(t_i) = \mathbf{P}^T \mathbf{f}_i\left(\mathbf{P}\mathbf{r}(t_{i-1}), \boldsymbol{\theta}\right)$$
$$\mathbf{y}(t_i) = \mathbf{h}_i\left(\mathbf{P}\mathbf{r}(t_i), \boldsymbol{\theta}\right)$$

whose trajectories $\mathbf{r}(t_i) = \mathbf{P}^T \mathbf{x}(t_i)$ evolve in -dimensional space